

Solution to the 3-Loop Φ -Derivable Approximation for Massless Scalar Thermodynamics

Eric Braaten and Emmanuel Petitgirard

Physics Department, Ohio State University, Columbus OH 43210, USA

Abstract

We develop a systematic method for solving the 3-loop Φ -derivable approximation to the thermodynamics of the massless ϕ^4 field theory. The method involves expanding sum-integrals in powers of g^2 and m/T , where g is the coupling constant, m is a variational mass parameter, and T is the temperature. The problem is reduced to one with the single variational parameter m by solving the variational equations order-by-order in g^2 and m/T . At the variational point, there are ultraviolet divergences of order g^6 that cannot be removed by any renormalization of the coupling constant. We define a finite thermodynamic potential by truncating at 5th order in g and m/T . The associated thermodynamic functions seem to be perturbatively stable and insensitive to variations in the renormalization scale.

I. INTRODUCTION

The thermodynamic functions for massless relativistic field theories at high temperature T can be calculated as weak-coupling expansions in the coupling constant g . They have been calculated explicitly through order g^5 for the massless ϕ^4 field theory [1,2], for QED [3–5], and for nonabelian gauge theories [6–8]. Unless the coupling constant is tiny, the weak-coupling expansions are poorly convergent and very sensitive to the renormalization scale. This makes the weak-coupling expansion essentially useless as a quantitative tool: the expansion seems to be reliable only when the coupling constant is so small that the corrections to ideal gas behavior are negligibly small. The physical origin of the instability seems to be effects associated with screening and quasiparticles.

A possible solution to this instability problem is to reorganize the weak-coupling expansion within a variational framework. A variational approximation can be defined by a *thermodynamic potential* Ω that depends on a set of variational parameters m_i . The free energy and other thermodynamic functions are given by the values of Ω and its derivatives at the variational point where $\partial\Omega/\partial m_i = 0$. A variational approximation is systematically improvable if there is a sequence of successive approximations to Ω that reproduce the weak-coupling expansions of the thermodynamic functions to successively higher orders in g . One example of a systematically improvable variational approximation is *screened perturbation theory*, which involves a single variational mass parameter [9].

A variational approach can be useful only if the correct physics can be captured by appropriate choices of the variational parameters. Information about screening and quasiparticle effects is contained within the exact propagator of the field theory. The possibility that these effects are responsible for the instability of the weak-coupling expansion suggests the use of the propagator as a variational function. Such a variational formulation was constructed for nonrelativistic fermions by Luttinger and Ward [10] and by Baym [11] around 1960. It was generalized to relativistic field theories by Cornwall, Jackiw and Tomboulis [12]. In the case of a relativistic scalar field theory, the propagator has the form $[P^2 + \Pi(P)]^{-1}$,

where $\Pi(P)$ is the self-energy which depends on the momentum P . The thermodynamic potential has the form

$$\Omega_0[\Pi] = \frac{1}{2} \not\int_P \left[\log(P^2 + \Pi) - \frac{\Pi}{P^2 + \Pi} \right] + \Phi[\Pi], \quad (1)$$

where the interaction functional $\Phi[\Pi]$ can be expressed as a sum of 2-particle-irreducible diagrams. It is constructed so that the solution to the variational equation $\delta\Omega_0/\delta\Pi = 0$ is the exact self-energy, and the value of Ω_0 at the variational point is the exact free energy. We can obtain a systematically improvable variational approximation by truncating Φ at n 'th order in the loop expansion, where $n = 2, 3, \dots$. We refer to this approximation as the *n-loop Φ -derivable approximation*.

One remarkable feature of the 2-loop Φ -derivable approximation is that the expression for the entropy reduces at the variational point to the 1-loop expression. This was shown for QED by Vanderheyden and Baym [13]. It was generalized to QCD by Blaizot, Iancu, and Rebhan [14], who have used it as the basis for a quasiparticle model for the thermodynamics of the quark-gluon plasma. An alternative quasiparticle model motivated by the 2-loop Φ -derivable approximation for QCD has been developed by Peshier [15]. One disadvantage of these models is that they are based on special properties of the 2-loop Φ -derivable approximation. Thus unlike the Φ -derivable approximation itself, they are not systematically improvable.

Φ -derivable approximations are *conserving* approximations, which means that they are consistent with the conservation laws that follow from Noether's theorem. Baym showed that an approximation is conserving if and only if it is Φ -derivable for some functional Φ [11]. The fact that they are conserving may make Φ -derivable approximations particularly useful for nonequilibrium problems [16]. They have already proved to be useful for nonequilibrium problems in $1 + 1$ dimensions [17].

While the Φ -derivable approximation is easily formulated, it is not so easy to solve. If the solution for the self-energy $\Pi(P)$ is independent of the momentum P , the Φ -derivable approximation can be reduced to a single-parameter variational problem that is easily solved

numerically. An example is the 2-loop Φ -derivable approximation for the massless ϕ^4 field theory [18]. If the solution for $\Pi(P)$ depends on P , the variational equation is a nontrivial integral equation. What makes it complicated to solve is that it is really a coupled set of integral equations for the function $\Pi(2\pi nT, |\mathbf{p}|)$ corresponding to all the Matsubara frequencies $2\pi nT, n = 0, \pm 1, \pm 2, \dots$. An even more severe obstacle is that the thermodynamic potential has ultraviolet divergences that vanish at the variational point only if Φ is calculated to all orders. They do not vanish away from the variational point, and they do not vanish at the variational point if the loop expansion for Φ is truncated.

In this paper, we solve the 3-loop Φ -derivable approximation for a massless scalar field theory with a ϕ^4 interaction. In section II, we define the coupling constant for the massless ϕ^4 field theory and give the weak-coupling expansions for several thermodynamic quantities. In section III, we summarize the basic features of the Φ -derivable approach. In section IV, we solve the 2-loop Φ -derivable approximation. We show that if the thermodynamic potential is expressed in terms of the true coupling constant, it has ultraviolet divergences of order g^4 . In section V, we solve the 3-loop Φ -derivable approximation by expanding sum-integrals systematically in powers of g and m/T . We show that the thermodynamic potential has ultraviolet divergences of order g^6 . We construct a finite thermodynamic potential by adding a term proportional to the square of the variational equation and then truncating after terms of 5th order in g and m/T . It defines a stable approximation to the thermodynamic functions that is rather insensitive to the choice of renormalization scale. In the concluding section, we discuss the outlook for extending our methods to gauge theories.

II. DEFINING THE THEORY

We would like to solve the Φ -derivable approximation to the thermodynamics of a massless scalar field theory with a ϕ^4 interaction. The theory can be defined by choosing a regularization scheme and specifying the value of the bare coupling constant g_0 . The lagrangian for the theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{24} g_0^2 \phi^4. \quad (2)$$

We choose to use dimensional regularization in $4 - 2\epsilon$ space-time dimensions. The bare coupling constant g_0 then has dimensions $(\text{mass})^\epsilon$. We define the renormalized coupling constant $g(\mu)$ by the modified minimal subtraction ($\overline{\text{MS}}$) prescription with renormalization scale μ . For convenience, we also define $\alpha = g^2/(4\pi)^2$ and $\alpha_0 = g_0^2/(4\pi)^2$. The relation between the bare and renormalized coupling constants is

$$\alpha_0 \mu^{-2\epsilon} = \alpha + \frac{3}{2\epsilon} \alpha^2 + \left(\frac{9}{4\epsilon^2} - \frac{17}{12\epsilon} \right) \alpha^3 + \left[\frac{27}{8\epsilon^3} - \frac{119}{24\epsilon^2} + \left(\frac{145}{48} + 2\zeta(3) \right) \frac{1}{\epsilon} \right] \alpha^4 + \dots, \quad (3)$$

where $\zeta(z)$ is the Riemann zeta function. In the $\overline{\text{MS}}$ scheme, there is usually also a factor of $(4\pi e^{-\gamma})^{2\epsilon}$ on the left side. We prefer to absorb this factor into the measure of the dimensionally regularized integrals. The beta function for the running coupling constant $\alpha(\mu)$ has been calculated to five-loop order [19]. The first few terms can be obtained by differentiating (3) with respect to μ and using the fact that α_0 is independent of μ :

$$\mu \frac{d}{d\mu} \alpha = -2\epsilon \alpha + \beta(\alpha), \quad (4)$$

where the beta function is

$$\beta(\alpha) = 3\alpha^2 - \frac{17}{3}\alpha^3 + \left(\frac{145}{8} + 12\zeta(3) \right) \alpha^4 + \mathcal{O}(\alpha^5). \quad (5)$$

The thermodynamic functions for this theory are known to order g^5 [6,1,2]. The weak-coupling expansion for the free energy density is

$$\mathcal{F} = \mathcal{F}_{\text{ideal}} \left[1 - \frac{15}{2} \frac{\alpha}{6} + 60 \left(\frac{\alpha}{6} \right)^{3/2} + 135 \left(L - \frac{59}{45} + \frac{\gamma}{3} + \frac{4}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{2}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) \left(\frac{\alpha}{6} \right)^2 + 540 \left(2 \log \frac{\alpha}{6} - 3L - \frac{5}{2} + 4 \log 2 - \gamma + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \left(\frac{\alpha}{6} \right)^{5/2} + \mathcal{O}(\alpha^3 \log \alpha) \right], \quad (6)$$

where $\alpha = \alpha(\mu)$, $L = \log(\mu/4\pi T)$, γ is Euler's gamma constant, and $\mathcal{F}_{\text{ideal}}$ is the pressure of the ideal gas of a free massless boson:

$$\mathcal{F}_{\text{ideal}} = -\frac{\pi^2}{90} T^4. \quad (7)$$

The weak-coupling expansions for the pressure, entropy density, and energy density are given by:

$$\mathcal{P} = -\mathcal{F}, \quad (8)$$

$$\mathcal{S} = -\frac{\partial}{\partial T}\mathcal{F}, \quad (9)$$

$$\mathcal{E} = -T^2 \frac{\partial}{\partial T} \left(\frac{1}{T} \mathcal{F} \right), \quad (10)$$

where the partial derivatives in (9) and (10) are evaluated with $\alpha(\mu)$ and μ held fixed. The free energy density and other thermodynamic functions satisfy simple renormalization group equations:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] \mathcal{F} = 0. \quad (11)$$

If the power series expansion for $\beta(\alpha)$ and the weak-coupling expansion for \mathcal{F} are truncated at some order in α , then the renormalization group equation (11) is satisfied only up to higher orders in α .

The renormalization group can be used to improve the weak-coupling expansions of the thermodynamic functions. This can be accomplished for the free energy (6) simply by setting $\mu = c(2\pi T)$, where c is a constant, in the logarithm L and in $\alpha(\mu)$. In this case, there are alternative thermodynamic definitions of the entropy and energy density in which the partial derivatives in (9) and (10) are replaced by total derivatives with respect to T . For example, the thermodynamic entropy is defined by

$$\mathcal{S}_{\text{th}} = -\frac{d}{dT}\mathcal{F}. \quad (12)$$

It differs from the entropy (9) by a term proportional to the left side of (11):

$$\mathcal{S}_{\text{th}} - \mathcal{S} = -\frac{1}{\mu} \frac{d\mu}{dT} \left[\mu \frac{\partial}{\partial \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] \mathcal{F}. \quad (13)$$

The right side is nonzero if the expansions of β and \mathcal{F} in powers of α are truncated.

The poor convergence of the weak-coupling expansion is illustrated in Figure 1, which shows the free energy divided by that of the ideal gas as a function of $g(2\pi T)$. The solid

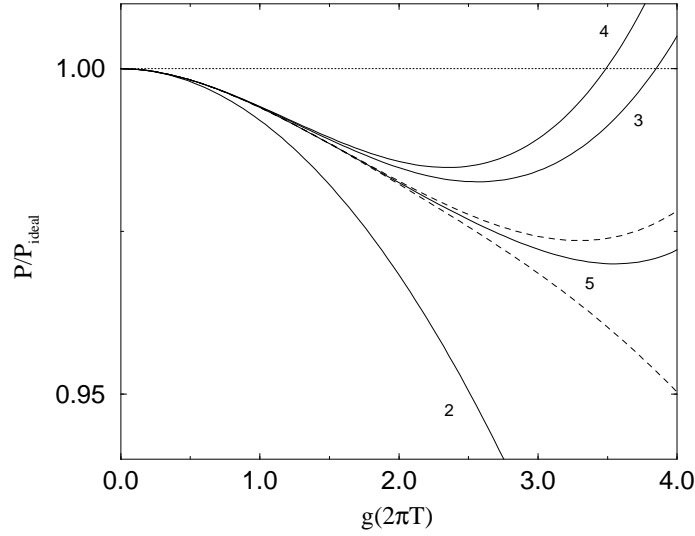


FIG. 1. Weak-coupling expansion for the pressure divided by that of the ideal gas as a function of $g(2\pi T)$. Solid lines correspond to the truncation at order g^n , $n = 2, 3, 4, 5$ for $\mu = 2\pi T$. Dashed lines correspond to the truncation at order g^5 for $\mu = \pi T$ and $\mu = 4\pi T$.

curves are the weak-coupling expansion with renormalization scale $\mu_T = 2\pi T$ truncated after orders g^n , $n = 2, 3, 4, 5$. They show no sign of converging except at extremely small values of g . The sensitivity of the weak-coupling expansion to the renormalization scale is also illustrated in Figure 1. The dashed lines are the weak-coupling expansion through order $g^5(\mu)$ for $\mu = \pi T$, $\mu = 4\pi T$ plotted as a function of $g(\mu_T)$. The relation between $g(\mu)$ and $g(\mu_T)$ is obtained by integrating numerically from $2\pi T$ to μ using the beta function (5) truncated after the α^4 term. Note that $g = 4$, which corresponds to $\alpha = 0.1013$, represents relatively strong coupling in the sense that it is close to the Landau pole. If $g(2\pi T) = 4$, the Landau pole at which the running coupling constant defined by the 3-loop beta function diverges is $\mu = 12.15(2\pi T)$.

Another observable that has been calculated to relatively high order is the screening mass [2]. It is known up to corrections of order g^5 :

$$m_s^2 = \frac{\alpha}{6}(2\pi T)^2 \left[1 - 6 \left(\frac{\alpha}{6} \right)^{1/2} + 6 \left[2 \log \frac{\alpha}{6} - 3L - 1 + 8 \log 2 - \gamma + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] \frac{\alpha}{6} + \mathcal{O}(\alpha^{3/2} \log \alpha) \right]. \quad (14)$$

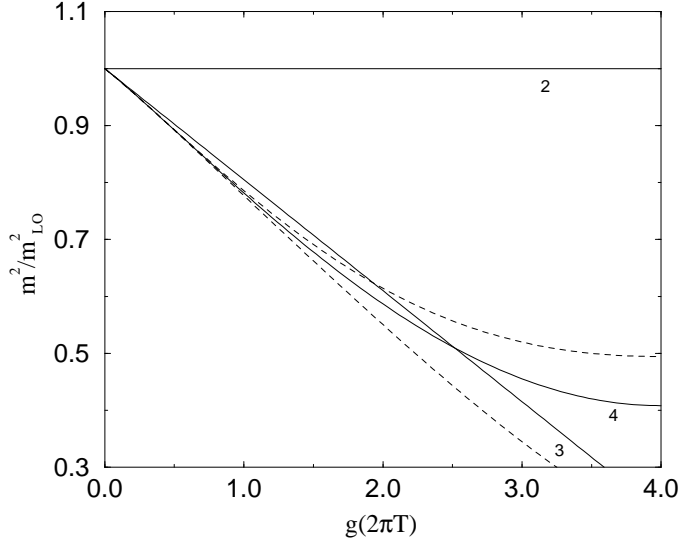


FIG. 2. Weak-coupling expansion for the screening mass divided by the leading order result $m_{\text{LO}}^2 = g^2(2\pi T)T^2/24$ as a function of $g(2\pi T)$. Solid lines correspond to the truncation at order g^n , $n = 2, 3, 4$ for $\mu = 2\pi T$. Dashed lines correspond to the truncation at order g^4 for $\mu = \pi T$ and $\mu = 4\pi T$.

The convergence of the weak-coupling expansion is illustrated in Figure 2, which shows m_s^2 divided by the leading order expression $m_{\text{LO}} = g^2(\mu_T)T^2/24$ as a function of $g(2\pi T)$. The solid lines are the weak-coupling expansion (14) truncated after the term of order g^n , $n = 2, 3, 4$. The weak-coupling expansions to orders g^3 and g^4 agree only at extremely small values of g . The expansion to order g^4 is also very sensitive to the renormalization scale as illustrated in Figure 2. The dashed lines show m_s^2/m_{LO}^2 , with m_s^2 truncated after the $g^4(\mu)$ term as a function of $g(2\pi T)$ for $\mu = \pi T$ and $\mu = 4\pi T$.

III. Φ -DERIVABLE APPROXIMATIONS

Φ -derivable approximations are based on the skeleton expansion of the free energy [10], which expresses it in terms of exact propagators. The exact propagator $D(P)$ for a massless scalar theory has the form

$$D(P) = \frac{1}{P^2 + \Pi(P)}. \quad (15)$$

The skeleton expansion for the free energy defines a functional $\Omega[D]$ of the propagator that we will refer to as the *thermodynamic potential*. The exact propagator satisfies the variational equation $\delta\Omega/\delta D = 0$. The free energy is therefore the variational minimum of the functional $\Omega[D]$:

$$\mathcal{F} = \Omega[D] \Big|_{\delta\Omega/\delta D=0}. \quad (16)$$

The thermodynamic potential can be expressed in the form

$$\Omega[D] = \frac{1}{2} \oint_P \log(P^2 + \Pi(P)) - \frac{1}{2} \oint_P \Pi(P) \frac{1}{P^2 + \Pi(P)} + \Phi[D], \quad (17)$$

where the functional $\Phi[D]$ contains all the effects of interactions. It can be expressed as a sum of 2-particle-irreducible Feynman diagrams. The variational equation $\delta\Omega/\delta D = 0$ can be written

$$\Pi(P) = 2 \frac{\delta\Phi[D]}{\delta D(P)}. \quad (18)$$

The variational derivative is calculated with the bare coupling constant α_0 and the ultraviolet cutoff parameter ϵ held fixed or, equivalently, with the renormalized coupling constant α and the renormalization scale μ held fixed.

An approximation is called Φ -derivable if it can be obtained by making the same approximation for the functional Φ in both (17) and (18) [11]. Φ -derivable approximations have several attractive features. One such feature is *self-consistency*: the thermodynamic potential (17), which includes the effects of interactions through the Φ term, is evaluated with a propagator that takes into account those same interactions through the variational equation (18). Another attractive feature is *thermodynamic consistency*. The entropy density can be expressed as

$$\mathcal{S} = \left(-\frac{\partial}{\partial T} \Omega[D] \right) \Big|_{\delta\Omega/\delta D=0}, \quad (19)$$

where $\partial/\partial T$ refers to the partial derivative with $D(P)$, α_0 , and ϵ held fixed. The thermodynamic entropy can be written

$$\mathcal{S}_{\text{th}} = -\frac{d}{dT} \left(\Omega[D] \Big|_{\delta\Omega/\delta D=0} \right), \quad (20)$$

where the derivative is evaluated with α_0 and ϵ fixed. The equivalence of (19) and (20) follows from the variational equation $\delta\Omega/\delta D = 0$.

The n -loop Φ -derivable approximations defined by truncating Φ at successively higher orders in the bare coupling constant, or equivalently the loop expansion, are particularly useful. They define a systematically improvable variational approximation. If Φ is truncated at order $g_0^{2(n-1)}$, the free energy will after renormalization include correctly all terms in the weak-coupling expansion through order g^{2n-1} , as well as an infinite series of higher terms in g . Thus each successive approximation will have an error that is suppressed parametrically by one higher order in g^2 . The propagator for each approximation is given by the self-consistent solution to a variational equation that takes into account the screening and quasiparticle effects characteristic of a plasma. Thus if the poor convergence of the weak-coupling expansion is due to treating plasma effects as a perturbation, the n -loop Φ -derivable approximations could provide the solution to this problem.

Unfortunately, Φ -derivable approximations have a problem that has limited their usefulness in relativistic field theories. The problem is that the thermodynamic potential Ω is generally ultraviolet divergent. There are two classes of ultraviolet divergences. One class consists of ultraviolet divergences that vanish at the variational point. They provide a serious complication for devising numerical methods for minimizing the thermodynamical potential. The other class of ultraviolet divergences are those that survive at the variational point. They arise from having truncated Φ at n^{th} order in the loop expansion. Renormalization theory guarantees that the weak-coupling expansion for the free energy will be finite through order g^{2n-1} , but there will generally be ultraviolet divergences at order g^{2n} . In the 2-loop Φ -derivable approximation for the massless ϕ^4 field theory, these ultraviolet divergences can be hidden by introducing a coupling constant that has the wrong beta func-

tion. However in the 3-loop Φ -derivable approximation, these ultraviolet divergences must be dealt with.

IV. TWO-LOOP Φ -DERIVABLE APPROXIMATION

In the 2-loop Φ -derivable approximation, the solution for the exact self-energy is independent of the momentum. We anticipate this fact by denoting the exact self-energy by m^2 . The variable m can be interpreted as the screening mass. The solution to the 2-loop Φ -derivable approximation has been given previously in Refs. [18] and [14]. We first solve the 2-loop Φ -derivable approximation exactly in terms of a coupling constant $\bar{g}(\mu)$ that has the wrong beta function. We show that the expansion of the thermodynamic potential in powers of m/T converges much more rapidly to the exact answer than the weak-coupling expansion in powers of \bar{g} . We then show that when the solution is expressed in terms of the true coupling constant g , there are unavoidable ultraviolet divergences at order g^4 .

The thermodynamic potential in the 2-loop Φ -derivable approximation is

$$\mu^{2\epsilon}\Omega_0(m) = \frac{1}{2}\oint_P \log(P^2 + m^2) - \frac{1}{2}m^2\oint_P \frac{1}{P^2 + m^2} + \frac{1}{8}(g_0\mu^{-\epsilon})^2 \left(\oint_P \frac{1}{P^2 + m^2} \right)^2, \quad (21)$$

where g_0 is the bare coupling constant. Our definition of the dimensionally regularized sum-integral is given in (B.1) and includes a factor of $\mu^{2\epsilon}$ in the measure. Each term in (21) has an overall multiplicative factor of $\mu^{2\epsilon}$, and therefore $\Omega_0(m)$ is independent of μ for fixed g_0 and ϵ . This factor of $\mu^{2\epsilon}$ is convenient, because it gives every term in (21) the dimensions (mass)⁴, even for $\epsilon \neq 0$. The variational equation $\partial\Omega/\partial m = 0$ reduces to the simple gap equation

$$m^2 = \frac{1}{2}(g_0\mu^{-\epsilon})^2 \oint_P \frac{1}{P^2 + m^2}. \quad (22)$$

The 2-loop Φ -derivable approximation for the free energy density \mathcal{F} is obtained by inserting the solution to the gap equation (22) into the thermodynamic potential (21).

A. Finite thermodynamic potential

The sum-integrals that appear in the thermodynamic potential (21) and in the gap equation (22) are ultraviolet divergent. Ideally, we would like to have a thermodynamic potential $\Omega(m)$ that is a finite function of the variational parameter m . To construct such a function, we begin by making the ultraviolet divergences explicit as poles in ϵ . The sum-integrals can be expressed in the form

$$\oint_P \log(P^2 + m^2) = -\frac{\pi^2}{45} T^4 \hat{\mu}^{2\epsilon} \left\{ a(\epsilon) - 15f(\hat{m}^2, \epsilon) + 45 \left[\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right] \hat{m}^{4-2\epsilon} \right\}, \quad (23)$$

$$\oint_P \frac{1}{P^2 + m^2} = \frac{1}{12} T^2 \hat{\mu}^{2\epsilon} \left\{ f'(\hat{m}^2, \epsilon) - 3(2 - \epsilon) \left(\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right) \hat{m}^{2-2\epsilon} \right\}, \quad (24)$$

where $\hat{m} = m/(2\pi T)$, $\hat{\mu} = \mu/(2\pi T)$, and $a(\epsilon)$ and $b(\epsilon)$ are constants that approach 1 in the limit $\epsilon \rightarrow 0$:

$$a(\epsilon) = \frac{e^{\gamma\epsilon} (2\pi)^{2\epsilon} \Gamma(\frac{5}{2}) \Gamma(4 - 2\epsilon) \zeta(4 - 2\epsilon)}{\Gamma(\frac{5}{2} - \epsilon) \Gamma(4) \zeta(4)}, \quad (25)$$

$$b(\epsilon) = \frac{4}{3} \left[\frac{e^{\gamma\epsilon} \Gamma(1 + \epsilon)}{\epsilon(1 - \epsilon)(2 - \epsilon)} - \frac{1}{2\epsilon} \right]. \quad (26)$$

The function $f(\hat{m}^2, \epsilon)$ is given by the integral

$$f(\hat{m}^2, \epsilon) = -16 \frac{e^{\gamma\epsilon} \Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} - \epsilon)} \int_0^\infty dx \left[\frac{x^{4-2\epsilon}}{(x^2 + \hat{m}^2)^{1/2}} \frac{1}{e^{2\pi(x^2 + \hat{m}^2)^{1/2}} - 1} - \frac{x^{3-2\epsilon}}{e^{2\pi x} - 1} \right], \quad (27)$$

and $f'(\hat{m}^2, \epsilon)$ is its derivative with respect to \hat{m}^2 . In the limit $\hat{m} \rightarrow 0$, $f(\hat{m}, 0)$ approaches \hat{m}^2 . The sum-integral (24) can be obtained simply by differentiating (23) with respect to m^2 . Inserting (23) and (24) into (21), the thermodynamic potential reduces to

$$\begin{aligned} \frac{\mu^{2\epsilon} \Omega_0(m)}{\mathcal{F}_{\text{ideal}}} &= \left[a(\epsilon) - 15f(\hat{m}^2, \epsilon) + 15\hat{m}^2 f'(\hat{m}^2, \epsilon) \right] \hat{\mu}^{2\epsilon} - 45(1 - \epsilon) \left(\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right) \hat{\mu}^{2\epsilon} \hat{m}^{4-2\epsilon} \\ &\quad - \frac{5}{4} \alpha_0 \mu^{-2\epsilon} \left[f'(\hat{m}^2, \epsilon) - 3(2 - \epsilon) \left(\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right) \hat{m}^{2-2\epsilon} \right]^2 \hat{\mu}^{4\epsilon}. \end{aligned} \quad (28)$$

The gap equation (22) reduces to

$$\hat{m}^2 = \frac{\alpha_0 \mu^{-2\epsilon}}{6} \left[f'(\hat{m}^2, \epsilon) - 3(2 - \epsilon) \left(\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right) \hat{m}^{2-2\epsilon} \right] \hat{\mu}^{2\epsilon}. \quad (29)$$

We wish to construct a finite thermodynamic potential Ω that has the same value at the variational point as (28). The gap equation (29) can be expressed in the form $\hat{m}^2 - \hat{G} = 0$. We can add any multiple of $(\hat{m}^2 - \hat{G})^2$ to (28) without changing its value at the variational point. We can use this freedom to cancel the double pole in ϵ in (28). Adding $45(\hat{m}^2 - \hat{G})^2/(\alpha_0\mu^{-2\epsilon})$ to (28), the resulting function reduces to

$$\frac{\mu^{2\epsilon}\Omega(m)}{\mathcal{F}_{\text{ideal}}} = \left[a(\epsilon) - 15f(\hat{m}^2, \epsilon) \right] \hat{\mu}^{2\epsilon} + \frac{45}{\alpha_0\mu^{-2\epsilon}} \hat{m}^4 + 45 \left(\frac{1}{2\epsilon} + \frac{3}{4}b(\epsilon) \right) \hat{\mu}^{2\epsilon} \hat{m}^{4-2\epsilon} \quad (30)$$

This thermodynamic potential will be finite as $\epsilon \rightarrow 0$ provided that the “renormalized coupling constant” $\bar{\alpha}(\mu)$ defined by

$$\frac{1}{\bar{\alpha}(\mu)} = \frac{1}{\alpha_0\mu^{-2\epsilon}} + \frac{1}{2\epsilon} \quad (31)$$

is finite as $\epsilon \rightarrow 0$. This equation does not define the value of $\bar{\alpha}(\mu)$ at any specific scale μ , because (31) does not have a smooth limit as $\epsilon \rightarrow 0$. However its dependence on μ is well defined. Differentiating (31) with respect to μ , we obtain

$$\mu \frac{d}{d\mu} \bar{\alpha}(\mu) = -2\epsilon \bar{\alpha}(\mu) + \bar{\alpha}^2(\mu). \quad (32)$$

This equation has a smooth limit as $\epsilon \rightarrow 0$, so the μ -dependence of $\bar{\alpha}(\mu)$ is well-defined. Note that the coefficient of $\bar{\alpha}^2$ in (32) differs from the coefficient of α^2 in the true beta function (5) by a factor of 3.

Using (31) to eliminate α_0 from (30) in favor of $\bar{\alpha}(\mu)$, we obtain a finite thermodynamic potential in the limit $\epsilon \rightarrow 0$:

$$\frac{\Omega(m)}{\mathcal{F}_{\text{ideal}}} = 1 - 15 \left[f(\hat{m}^2) - 3 \left(\frac{1}{\bar{\alpha}} + \log \frac{\mu}{m} + \frac{3}{4} \right) \hat{m}^4 \right], \quad (33)$$

where the function $f(\hat{m}^2)$ is now the limit as $\epsilon \rightarrow 0$ of (27):

$$f(\hat{m}^2) = -16 \int_0^\infty dx \left(\frac{x^4}{\sqrt{x^2 + \hat{m}^2}} \frac{1}{e^{2\pi(x^2 + \hat{m}^2)^{1/2}} - 1} - \frac{x^3}{e^{2\pi x} - 1} \right). \quad (34)$$

Upon varying (33) with respect to m^2 , we obtain the gap equation

$$\hat{m}^2 = \frac{\bar{\alpha}}{6} \left[f'(\hat{m}^2) - 6 \left(\log \frac{\mu}{m} + \frac{1}{2} \right) \hat{m}^2 \right]. \quad (35)$$

The free energy $\mathcal{F}(T)$ in the 2-loop Φ -derivable approximation is obtained by solving (35) for \hat{m} and inserting the solution into (33).

The thermodynamic potential (33) is independent of the scale μ , because it satisfies the renormalization group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \bar{\alpha}^2 \frac{\partial}{\partial \bar{\alpha}} \right] \Omega(m) = 0. \quad (36)$$

However the Φ -derivable free energy still has an ambiguity associated with the renormalization scale. The ambiguity arises because (31) determines the dependence of $\bar{\alpha}(\mu)$ on μ but it does not define the value of $\bar{\alpha}(\mu)$ at any scale μ . The value of $\bar{\alpha}(\mu)$ becomes well-defined only after some initial value $\bar{\alpha}(\mu_0)$ is specified. A convenient choice for μ_0 is the scale at which $\bar{\alpha}$ has the same value as the true coupling constant α :

$$\bar{\alpha}(\mu_0) = \alpha(\mu_0). \quad (37)$$

Since $\bar{\alpha}$ and α have different beta functions, their values are different at other values of μ :

$$\bar{\alpha}(\mu) = \alpha(\mu) - 2 \log \frac{\mu}{\mu_0} \alpha^2(\mu) + \left[4 \log^2 \frac{\mu}{\mu_0} + \frac{17}{3} \log \frac{\mu}{\mu_0} \right] \alpha^3(\mu) + \mathcal{O}(\alpha^4). \quad (38)$$

The ambiguity in the matching scale μ_0 is the price that must be paid for using a coupling constant with the wrong beta function. A reasonable choice for the matching scale is $\mu_0 = 2\pi T$. However in estimating the errors of the Φ -derivable approximation, one should take into account the ambiguity in μ_0 , perhaps by allowing it to vary by a factor of 2 around the preferred value.

B. Weak-coupling Expansion

The two-loop free energy in the Φ -derivable approximation can be expanded systematically as a weak-coupling expansion in powers of $\bar{\alpha}^{1/2}$. One can obtain a sequence of analytic approximation to the free energy by truncating the expansion. Unfortunately we will find that the series converges rather slowly.

To express the free energy as an expansion in powers of $\bar{\alpha}^{1/2}$, we need to expand the solution to the gap equation (35) in powers of $\bar{\alpha}^{1/2}$ and insert it into (33). We first need to expand the function $f(\hat{m}^2)$ in powers of \hat{m} . The expansion, which is derived in Appendix A, has the form

$$f(\hat{m}^2) = \hat{m}^2 - 4\hat{m}^3 - 3 \left(\log \frac{\hat{m}}{2} - \frac{3}{4} + \gamma \right) \hat{m}^4 + \sum_{n=3}^{\infty} f_n \hat{m}^{2n}, \quad (39)$$

where the coefficients f_n are

$$f_n = (-1)^{n+1} \frac{6\Gamma(n - \frac{3}{2})\zeta(2n - 3)}{\Gamma(\frac{1}{2})n!}. \quad (40)$$

Note that the only odd power of \hat{m} is the \hat{m}^3 term. The asymptotic behavior of the coefficients as $n \rightarrow \infty$ is $f_n \rightarrow (-1)^{n+1}(6/\sqrt{\pi})n^{-5/2}$. Using the ratio test, we can infer that the region of convergence of the power series in (39) is $|m| < 2\pi T$.

Inserting the expansion (39) into (33), the thermodynamic potential becomes

$$\frac{\Omega(m)}{\mathcal{F}_{\text{ideal}}} = 1 - 15 \left\{ \hat{m}^2 - 4\hat{m}^3 - 3 \left(\frac{1}{\bar{\alpha}} + L + \gamma \right) \hat{m}^4 + \sum_{n=3}^{\infty} f_n \hat{m}^{2n} \right\}, \quad (41)$$

where $L = \log(\mu/4\pi T)$. Differentiating the expansion (39) and inserting it into (35), the gap equation becomes

$$\hat{m}^2 = \frac{\bar{\alpha}}{6} \left\{ 1 - 6\hat{m} - 6(L + \gamma)\hat{m}^2 + \sum_{n=3}^{\infty} n f_n \hat{m}^{2n-2} \right\}. \quad (42)$$

Using the ratio test, we can infer that the regions of convergence of the series in (41) and (42) is $|m| < 2\pi T$. As long as m is inside this radius of convergence, the functions of m/T on the right can be evaluated simply by adding terms in the series until the desired numerical accuracy is achieved. The first few terms in the series for (42) are

$$\hat{m}^2 = \frac{\bar{\alpha}}{6} \left\{ 1 - 6\hat{m} - 6(L + \gamma)\hat{m}^2 + 1.803\hat{m}^4 - 0.778\hat{m}^6 + \dots \right\}. \quad (43)$$

For asymptotically weak coupling, the solution is $m = \bar{g}T/\sqrt{24}$. The radius of convergence in m translates into $|\bar{g}| < 4\pi\sqrt{6} \simeq 30.8$.

The solution to the gap equation (42) up to errors of order $\bar{\alpha}^{5/2}$ is

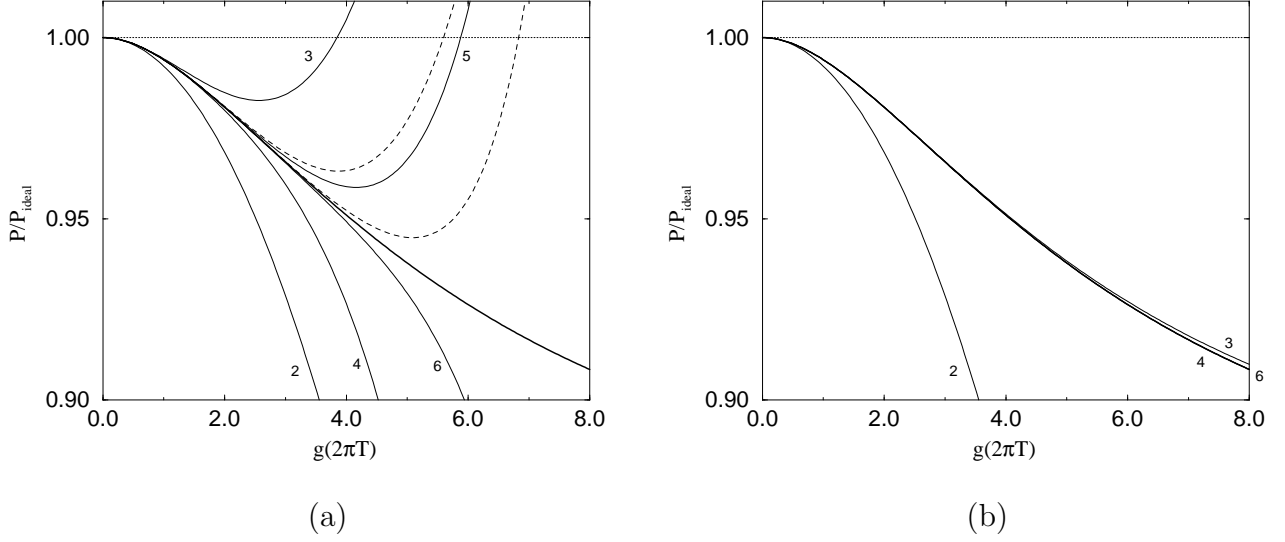


FIG. 3. 2-loop Φ -derivable approximation to the pressure divided by that of the ideal gas (heavy solid line) as a function of $\bar{g}(2\pi T)$. In Figure (a), this is compared to the weak-coupling expansion truncated at orders g^n , $n = 2, 3, 4, 5, 6$. In Figure (b), it is compared to the m/T expansion truncated at orders g^n , $n = 2, 3, 4, 6$.

$$\hat{m}^2 = \frac{\bar{\alpha}}{6} \left\{ 1 - 6 \left(\frac{\bar{\alpha}}{6} \right)^{\frac{1}{2}} - 6(L + \gamma - 3) \frac{\bar{\alpha}}{6} + 27(2L + 2\gamma - 1) \left(\frac{\bar{\alpha}}{6} \right)^{3/2} + 36 \left(L^2 - (6 - 2\gamma)L - 3.080 \right) \left(\frac{\bar{\alpha}}{6} \right)^2 + \mathcal{O}(\bar{\alpha}^{5/2}) \right\}, \quad (44)$$

where $\bar{\alpha} = \bar{\alpha}(\mu)$ and $L = \log(\mu/4\pi T)$. The weak-coupling expansion for the two-loop Φ -derivable free energy is obtained by inserting the expansion (44) into (41) and expanding in powers of $\bar{\alpha}^{1/2}$. The expansion up to errors of order $\bar{\alpha}^{7/2}$ is

$$\frac{1}{15} \left(\frac{\mathcal{F}}{\mathcal{F}_{\text{ideal}}} - 1 \right) = -\frac{1}{2} \frac{\bar{\alpha}}{6} + 4 \left(\frac{\bar{\alpha}}{6} \right)^{3/2} + 3(L + \gamma - 6) \left(\frac{\bar{\alpha}}{6} \right)^2 - 18(2L + 2\gamma - 3) \left(\frac{\bar{\alpha}}{6} \right)^{5/2} - 18 \left(L^2 - (12 - 2\gamma)L - 0.560 \right) \left(\frac{\bar{\alpha}}{6} \right)^3 + \mathcal{O}(\bar{\alpha}^{7/2}). \quad (45)$$

The convergence behavior of the weak-coupling expansion for the 2-loop Φ -derivable approximation is illustrated in Figure 3, which shows the free energy divided by that of the ideal gas as a function of $\bar{g}(2\pi T)$. The exact result is shown as a heavy solid line. The light solid lines show the weak-coupling expansions for $\mu = 2\pi T$ truncated after the terms

of order \bar{g}^n , $n = 2, 3, 4, 5$. The weak-coupling expansion converges to the exact answer, which is reflected in the fact that successive approximations track the exact answer out to higher values of \bar{g} . However the convergence is relatively slow. While the exact answer is completely independent of the renormalization scale μ , the weak-coupling expansion has a strong sensitivity to μ as illustrated in Figure 3a. The dashed lines are the weak-coupling expansions for $\mu = \pi T$ and $\mu = 4\pi T$ truncated after the $\bar{g}^5(\mu)$ term as a function of $\bar{g}(2\pi T)$. The relation between $\bar{g}(\mu)$ and $\bar{g}(2\pi T)$ is given by the exact solution to the renormalization group equation:

$$\bar{\alpha}(\mu) = \frac{\bar{\alpha}(2\pi T)}{1 - \bar{\alpha}(2\pi T) \log(\mu/2\pi T)}. \quad (46)$$

Note that the Landau pole appears at $\mu = e^{1/\bar{\alpha}(2\pi T)}(2\pi T)$. For $\bar{g}(2\pi T) = 8$, it appears at $\mu = 11.79(2\pi T)$. Thus $\bar{g} = 8$ in the 2-loop Φ -derivable approximation is roughly as close to the Landau pole as $g = 4$, where g is the running coupling constant defined by the 3-loop beta function of the ϕ^4 theory.

C. Expansion in m/T

The slow convergence of the weak-coupling expansion provides motivation for developing a better sequence of successive approximations to the Φ -derivable free energy. We would like an approximation scheme that is applicable not only to the two-loop Φ -derivable approximation, but can be extended to higher loops. We will develop an approximation scheme that is based on expanding sum-integrals in powers of m/T .

The expansion of the finite thermodynamic potential $\Omega(m)$ in powers of $\hat{m} = m/(2\pi T)$ is given in (41). We can define a series of successive approximations $\Omega^{(n)}(m)$ by truncating after terms of n -th order in \bar{g} , with m treated as order \bar{g} . For example, the thermodynamic potential truncated after the term of order \bar{g}^6 is

$$\frac{1}{15} \left(\frac{\Omega^{(6)}(m)}{\mathcal{F}_{\text{ideal}}} - 1 \right) = \frac{3\hat{m}^4}{\bar{\alpha}} - \hat{m}^2 + 4\hat{m}^3 + 3(L + \gamma)\hat{m}^4 - \frac{1}{2}\zeta(3)\hat{m}^6. \quad (47)$$

The corresponding gap equation is

$$\hat{m}^2 = \frac{\bar{\alpha}}{6} \left[1 - 6\hat{m} - 6(L + \gamma)\hat{m}^2 + \frac{3}{2}\zeta(3)\hat{m}^4 \right]. \quad (48)$$

The first few of these successive approximations to the free energy can be calculated analytically. The $n = 2$ truncation includes only the first two terms of the right side of (47). The solution to the gap equation is simply $\hat{m} = \sqrt{\bar{\alpha}/6}$. The resulting expression for the free energy is

$$\frac{\mathcal{F}^{(2)}}{\mathcal{F}_{\text{ideal}}} = 1 - \frac{5}{4}\bar{\alpha}. \quad (49)$$

The $n = 3$ truncation includes also the \hat{m}^3 term in (47). The gap equation is a quadratic equation whose solution is

$$\hat{m} = \sqrt{\frac{1}{6}\bar{\alpha} + \frac{1}{4}\bar{\alpha}^2} - \frac{1}{2}\bar{\alpha}. \quad (50)$$

The resulting expression for the free energy is

$$\frac{\mathcal{F}^{(3)}}{\mathcal{F}_{\text{ideal}}} = 1 - \frac{5}{4}\bar{\alpha} \left[1 + 6\bar{\alpha} + 6\bar{\alpha}^2 - 4(2 + 3\bar{\alpha})\sqrt{\frac{\bar{\alpha}}{6} + \frac{\bar{\alpha}^2}{4}} \right]. \quad (51)$$

In the strong coupling limit $\bar{\alpha} \rightarrow \infty$, the solutions for the $n = 3$ truncation have finite limits: $\hat{m} \rightarrow \frac{1}{6}$ and $\mathcal{F}^{(3)}/\mathcal{F}_{\text{ideal}} \rightarrow \frac{31}{36}$. Note that solutions (50) and (51) both depend on the renormalization scale μ through $\bar{\alpha}(\mu)$. The screening mass and the free energy $\mathcal{F}^{(4)}$ for the $n = 4$ truncation can be obtained simply by substituting $\bar{\alpha} \rightarrow 1/(1/\bar{\alpha} + L + \gamma)$ into (50) and (51). The resulting expressions are completely independent of μ .

The thermodynamic functions defined by truncating the m/T expansion converge much more quickly than the weak-coupling expansion. This is illustrated in Fig. 3b, which shows the free energy divided by the ideal gas as a function of $\bar{g}(2\pi T)$. The heavy solid line is the exact 2-loop Φ -derivable result. The slighter solid lines are the results from truncating the thermodynamic potential after terms of order g^n , $n = 2, 3, 4, 6$, with m treated as order g . For $n = 2, 3$, we set $\mu = 2\pi T$, but beginning at order g^4 , the result is completely independent of μ . The results for $n = 4, 6$ are so close to the exact result that they cannot be distinguished from it in the Figure. Thus the convergence of the m/T expansion is remarkably fast.

D. Solution in terms of True Coupling Constant

In the 3-loop Φ -derivable approximation, we will not have the luxury of being able to calculate the thermodynamic potential as explicitly as in (28). The best we will be able to do is calculate the sum-integrals as truncated expansions in m/T . We will also be unable to absorb all the ultraviolet divergences in the free energy into a renormalized coupling constant $\bar{\alpha}(\mu)$ that runs differently from the true coupling constant $\alpha(\mu)$. We will therefore repeat the solution to the 2-loop Φ -derivable approximation in a manner that is as parallel as possible to the method we will use for the 3-loop case.

Our goal will be to construct a thermodynamic potential that depends on the true coupling constant $\alpha(\mu)$ defined by (3) and is finite to as high an order in g as is possible. We have the freedom to remove ultraviolet divergences from the thermodynamic potential (21) without changing the free energy by adding multiples of $(m^2 - G)^2$, where G is the right side of the gap equation (22). We can cancel the divergences proportional to g_0^2/ϵ^2 in the last term of (21) by subtracting $(m^2 - G)^2/(2(g_0\mu^{-\epsilon})^2)$. The resulting thermodynamic potential is

$$\mu^{2\epsilon}\Omega(m) = \frac{1}{2} \not\int_P \log(P^2 + m^2) - \frac{m^4}{2(g_0\mu^{-\epsilon})^2}. \quad (52)$$

Expanding in powers of m/T , we get

$$\frac{\mu^{2\epsilon}\Omega(m)}{\mathcal{F}_{\text{ideal}}} = 1 - 15 \left\{ \hat{m}^2 - 4\hat{m}^3 - 3 \left(\frac{1}{\alpha_0\mu^{-2\epsilon}} + \frac{1}{2\epsilon} + L + \gamma \right) \hat{m}^4 \right\} + \mathcal{O}(\hat{m}^6). \quad (53)$$

If we substitute the expression (3) for the bare coupling constant in terms of the true renormalized coupling constant $\alpha(\mu)$, we find that there are still ultraviolet divergences proportional to \hat{m}^4 :

$$\frac{\mu^{2\epsilon}\Omega(m)}{\mathcal{F}_{\text{ideal}}} = 1 - 15 \left\{ \hat{m}^2 - 4\hat{m}^3 - 3 \left[\frac{1}{\alpha} + \left(-\frac{1}{\epsilon} + L + \gamma \right) - \frac{17}{12\epsilon}\alpha \right] \hat{m}^4 \right\} + \mathcal{O}(g^6). \quad (54)$$

The closest we can come to constructing a finite thermodynamic potential that is a function of the true coupling constant is to truncate after the terms of order g^3 :

$$\frac{1}{15} \left(\frac{\Omega(m)}{\mathcal{F}_{\text{ideal}}} - 1 \right) = \frac{3}{\alpha} \hat{m}^4 - \hat{m}^2 + 4\hat{m}^3. \quad (55)$$

The corresponding gap equation is

$$\hat{m}^2 = \frac{\alpha}{6} [1 - 6\hat{m}]. \quad (56)$$

Note that these expressions differ from those obtained by truncating (47) and (48), because they involve the true coupling constant α instead $\bar{\alpha}$. The solution to the gap equation (56) is obtained by substituting α for $\bar{\alpha}$ in (50). The resulting expression for the free energy is obtained by substituting α for $\bar{\alpha}$ in (51).

V. THREE-LOOP Φ -DERIVABLE APPROXIMATION

In the three-loop Φ -derivable approximation, the solution to the variational equation (18) for the self-energy $\Pi(P)$ is a nontrivial function of the momentum P . The thermodynamic potential in the 3-loop Φ -derivable approximation is

$$\begin{aligned} \mu^{2\epsilon}\Omega &= \frac{1}{2} \not\!\!\!\int_P \log(P^2 + \Pi(P)) - \frac{1}{2} \not\!\!\!\int_P \frac{\Pi(P)}{P^2 + \Pi(P)} \\ &\quad + \frac{1}{8} (g_0 \mu^{-\epsilon})^2 \left(\not\!\!\!\int_P \frac{1}{P^2 + \Pi(P)} \right)^2 - \frac{1}{48} (g_0 \mu^{-\epsilon})^4 \mathfrak{S}_{\text{ball}}, \end{aligned} \quad (57)$$

where $\mathfrak{S}_{\text{ball}}$ is the basketball sum-integral:

$$\mathfrak{S}_{\text{ball}} = \not\!\!\!\int_{PQR} \frac{1}{P^2 + \Pi(P)} \frac{1}{Q^2 + \Pi(Q)} \frac{1}{R^2 + \Pi(R)} \frac{1}{S^2 + \Pi(S)}, \quad (58)$$

where $S = -(P + Q + R)$. The variational equation obtained by varying (57) with respect to $\Pi(P)$ is

$$\Pi(P) = \frac{1}{2} (g_0 \mu^{-\epsilon})^2 \not\!\!\!\int_Q \frac{1}{Q^2 + \Pi(Q)} - \frac{1}{6} (g_0 \mu^{-\epsilon})^4 \mathfrak{S}_{\text{sun}}(P), \quad (59)$$

where $\mathfrak{S}_{\text{sun}}(P)$ is the sunset sum-integral:

$$\mathfrak{S}_{\text{sun}}(P) = \not\!\!\!\int_{QR} \frac{1}{Q^2 + \Pi(Q)} \frac{1}{R^2 + \Pi(R)} \frac{1}{S^2 + \Pi(S)}. \quad (60)$$

The three-loop Φ -derivable approximation to the free energy is obtained by solving (59) for $\Pi(P)$ and inserting the solution into the thermodynamic potential (57).

A. Mass Variable

Our strategy for solving the 3-loop Φ -derivable approximation is to introduce a mass variable m which is of order gT in the weak-coupling limit and then calculate the sum-integrals in (57) and (59) as double expansions in α_0 and m/T . We will find that the gap equation (59) for $\Pi(P)$ has a recursive structure that allows us to solve for its dependence on P . Inserting the solution into (57), we can reduce the thermodynamic potential $\Omega(m)$ to a double expansion in α_0 and m/T .

The simplest choice for the variational parameter conceptually is the self-energy at zero external momentum $\Pi(0)$. This is simply the value of the variational function $\Pi(P)$ at a particular point in momentum space. One of the disadvantages of $\Pi(0)$ as a variational parameter is that it is ultraviolet divergent. Renormalization theory guarantees that in conventional perturbation theory the ultraviolet divergences in the exact propagator $1/(P^2 + \Pi(P))$ are canceled order-by-order in the renormalized coupling constant α by multiplying by Z^{-1} , where Z is the wavefunction renormalization constant. Thus $Z\Pi(0)$ should be finite order-by-order in α . Since Z is divergent at order α^2 and $\Pi(0)$ begins at order α , $\Pi(0)$ must have ultraviolet divergences at order α^3 . Now the Φ -derivable approximation involves a truncation of perturbation theory, so multiplicative renormalizability is guaranteed only up to the truncation error. Since the three-loop Φ -derivable approximation does not include all three-loop self-energy diagrams, the truncation error for $Z\Pi(0)$ is of order α^3 . If we use $\Pi(0)$ as a variational parameter, ultraviolet divergences in the solution to the gap equation will come partly from true divergences of $\Pi(0)$ and partly from the failure of multiplicative renormalizability in the 3-loop Φ -derivable approximation.

A choice for the variational parameter with more convenient ultraviolet properties is the screening mass defined by

$$\Pi(0, \mathbf{p}) \rightarrow m_s^2 \quad \text{as } p^2 \rightarrow -m_s^2. \quad (61)$$

Renormalization theory guarantees that in the conventional perturbation theory m_s^2 is finite

order-by-order in the renormalized coupling constant α . Since the three-loop Φ -derivable approximation does not include all three-loop self-energy diagrams, m_s^2 is guaranteed to be finite only up to corrections of order α^3 . But any divergences at this order will come from the failure of multiplicative renormalizability in the 3-loop Φ -derivable approximation.

In the three-loop Φ -derivable approximation, the screening mass m_s defined by (61) is the solution to the gap equation

$$m^2 = \frac{1}{2}(g_0\mu^{-\epsilon})^2 \not\int_Q \frac{1}{Q^2 + \Pi(Q)} - \frac{1}{6}(g_0\mu^{-\epsilon})^4 \mathfrak{S}_{\text{screen}} , \quad (62)$$

where we have introduced the short-hand

$$\mathfrak{S}_{\text{screen}} \equiv \mathfrak{S}_{\text{sun}}(0, \mathbf{p}) \Big|_{p=im} . \quad (63)$$

Since the sum-integrals in (62) can be expressed as expansions in $\Pi(0)/T$, this equation defines m^2 as a function of $\Pi(0)$. We will find it more convenient to calculate the sum-integrals as expansions in m/T . Subtracting (62) from (59), the variational equation reduces to

$$\Pi(P) = m^2 - \frac{1}{6}(g_0\mu^{-\epsilon})^4 [\mathfrak{S}_{\text{sun}}(P) - \mathfrak{S}_{\text{screen}}] . \quad (64)$$

If we set $P = 0$ and evaluate the sunset sum-integrals on the right-side as expansions in m/T , we get an equation for $\Pi(0)$ as a function of m . This guarantees that the mass variable m defined by (61) can be used as a variational parameter in place of $\Pi(0)$. The solution to the gap equation (62) for m will be the Φ -derivable approximation to the screening mass m_s .

B. Solution to the Variational Equation

In the variational equation (64) for the self-energy $\Pi(P)$, the sum-integrals extend over all momenta. There are two important momentum scales: the *hard* scale $2\pi T$ and the *soft* scale m . The hard region for the momentum $P = (2\pi nT, \mathbf{p})$ includes $n \neq 0$ for all \mathbf{p} and

also $n = 0$ with \mathbf{p} of order T . The soft region is $n = 0$ and \mathbf{p} of order m . We will solve the variational equation in the two momentum regions separately.

In order to solve the variational equation for $\Pi(P)$, we will assume that the scales m and $2\pi T$ can be separated. This will allow $\Pi(P)$ to be expanded in powers of $g_0\mu^{-\epsilon}$ and m/T . In the variational equation (64), the bare coupling constant appears only in the combination $(g_0\mu^{-\epsilon})^4$. We therefore assume that the self-energy $\Pi(P)$ can be expanded in powers of $(g_0\mu^{-\epsilon})^4$, with coefficients that are functions of P , m and T . These functions in turn can be expanded in powers of m/T . For hard momentum P , we write the expansion in the form

$$\begin{aligned} \Pi(P) = & m^2 + (g_0\mu^{-\epsilon})^4 [\Pi_{4,0}(P) + \Pi_{4,1}(P) + \Pi_{4,2}(P) + \Pi_{4,3}(P) + \dots] \\ & + (g_0\mu^{-\epsilon})^8 [\Pi_{8,-2}(P) + \Pi_{8,-1}(P) + \dots] + \dots \quad \text{for hard } P, \end{aligned} \quad (65)$$

where $\Pi_{n,k}(P)$ is of order $T^2(m/T)^k$ when P is of order T . We have anticipated the solution to the variational equation by writing explicitly all terms that contribute through order $g^7 T^2$. For soft momentum $P = (0, \mathbf{p})$, we write the self-energy as $\Pi(0, \mathbf{p}) = m^2 + \sigma(p)$. The m/T expansion of the function $\sigma(p)$ has the form

$$\begin{aligned} \sigma(p) = & (g_0\mu^{-\epsilon})^4 [\sigma_{4,-2}(p) + \sigma_{4,0}(p) + \sigma_{4,1}(p) + \dots] \\ & + (g_0\mu^{-\epsilon})^8 [\sigma_{8,-4}(p) + \dots] + \dots \quad \text{for soft } p, \end{aligned} \quad (66)$$

where $\sigma_{n,k}(p)$ is of order $m^2(m/T)^k$ when p is of order m . We have anticipated the solution to the variational equation by writing explicitly all terms that contribute through order $g^5 m^2$. Upon inserting the expansions (65) and (66) into the variational equation (64) and expanding systematically in powers of $g_0\mu^{-\epsilon}$ and m/T , we will find that it has a recursive structure that either determines $\Pi_{n,k}(P)$ and $\sigma_{n,k}(p)$ or allows them to be expressed in terms of lower order functions.

The m/T expansions for the sunset sum-integral $\mathfrak{S}_{\text{sun}}(P)$ are derived in Appendix A. For hard momentum P , the m/T expansion including all terms up to errors of order $g^4 T^2$ is the sum of (A.12), (A.14), and (A.16):

$$\begin{aligned}
\mathfrak{S}_{\text{sun}}(P) = & \oint_{QR} \frac{1}{Q^2 R^2 (P+Q+R)^2} + 3T I_1 \oint_Q \frac{1}{Q^2 (P+Q)^2} \\
& + 3T^2 I_1^2 \frac{1}{P^2} - 3m^2 \oint_{QR} \frac{1}{(Q^2)^2 R^2 (P+Q+R)^2} \\
& - 3T m^2 I_1 \oint_Q \left(\frac{1}{(Q^2)^2 (P+Q)^2} + \frac{(4/d)\mathbf{q}^2}{(Q^2)^3 (P+Q)^2} \right) \\
& - 3(g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{r}} \frac{\sigma_{4,-2}(r)}{(r^2 + m^2)^2} \oint_Q \frac{1}{Q^2 (P+Q)^2} + \mathcal{O}(g^4 T^2). \tag{67}
\end{aligned}$$

The momentum integral I_1 is proportional to $m^{1-2\epsilon}$ and is given in (C.7). We will find that the momentum integral involving $\sigma_{4,-2}$ is proportional to $m^{-3-4\epsilon}$. For soft momentum $P = (0, \mathbf{p})$, the sunset sum-integral, including all terms up to errors of order $g^4 T^2$, is the sum of (A.18), (A.20), and (A.22):

$$\begin{aligned}
\mathfrak{S}_{\text{sun}}(0, \mathbf{p}) = & T^2 I_{\text{sun}}(p) + 3T I_1 \oint_Q \frac{1}{(Q^2)^2} \\
& - (p^2 + 3m^2) \oint_{QR} \frac{1}{(Q^2)^2 R^2 (Q+R)^2} + \frac{4}{d} p^2 \oint_{QR} \frac{\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} \\
& - 3T(p^2 + m^2) I_1 \oint_Q \frac{1}{(Q^2)^3} + \frac{12}{d} T(p^2 - m^2) I_1 \oint_Q \frac{\mathbf{q}^2}{(Q^2)^4} \\
& - 3(g_0 \mu^{-\epsilon})^4 \left[T \int_{\mathbf{r}} \frac{\sigma_{4,-2}(r)}{(r^2 + m^2)^2} \oint_Q \frac{1}{(Q^2)^2} + T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) \right] \\
& + \mathcal{O}(g^4 T^2). \tag{68}
\end{aligned}$$

The sunset integral $I_{\text{sun}}(p)$ is defined in (C.10) and scales like $m^{-4-2\epsilon}$ when p is of order m . The bubble integral $I_{\text{bub}}(p)$ is defined in (C.9) and scales like $m^{-1-2\epsilon}$ when p is of order m .

We proceed to solve the variational equation (64) for $\Pi(P)$. We first consider the equation for hard momentum P . On the left side of (64), we insert the expansion (65) for $\Pi(P)$. On the right side of (64), we insert the m/T expansions (67) and (68) of the sunset sum-integrals. Matching coefficients of $(g_0 \mu^{-\epsilon})^{4n}$ on both sides of the equation and then identifying the terms in the m/T expansions, we find

$$\Pi_{4,0}(P) = -\frac{1}{6} \oint_{QR} \frac{1}{Q^2 R^2 (P+Q+R)^2} + \frac{1}{6} T^2 I_{\text{sun}}(im), \tag{69}$$

$$\Pi_{4,1}(P) = -\frac{1}{2} T I_1 \oint_Q \left(\frac{1}{Q^2 (P+Q)^2} - \frac{1}{(Q^2)^2} \right), \tag{70}$$

$$\Pi_{4,2}(P) = -\frac{1}{2} T^2 I_1^2 \frac{1}{P^2} + \frac{1}{2} m^2 \oint_{QR} \frac{1}{(Q^2)^2 R^2 (P+Q+R)^2}$$

$$-\frac{1}{3}m^2 \not\!\!\!\int_{QR} \left(\frac{1}{(Q^2)^2 R^2 (Q+R)^2} + \frac{(2/d)\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} \right), \quad (71)$$

$$\Pi_{8,-2} = -\frac{1}{2}T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2+m)^2} I_{\text{bub}}(|\mathbf{k}+\mathbf{q}|) \Big|_{k=im}. \quad (72)$$

We next consider the variational equation for soft momentum $P = (0, \mathbf{p})$. On the left side of (64), we insert the expansion (66) for $\Pi(0, \mathbf{p})$. On the right side of (64), we insert the m/T expansions (68) of the sunset sum-integral at soft momentum. Matching coefficients of $(g_0\mu^{-\epsilon})^{4n}$ on both sides of the equation and then identifying the terms in the m/T expansions, we find

$$\sigma_{4,-2}(p) = -\frac{1}{6}T^2 [I_{\text{sun}}(p) - I_{\text{sun}}(im)], \quad (73)$$

$$\sigma_{4,0}(p) = \frac{1}{6}(p^2 + m^2) \not\!\!\!\int_{QR} \left(\frac{1}{(Q^2)^2 R^2 (Q+R)^2} - \frac{(4/d)\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} \right), \quad (74)$$

$$\sigma_{8,-4}(p) = \frac{1}{2}T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2+m^2)^2} I_{\text{bub}}(|\mathbf{p}+\mathbf{q}|) + \Pi_{8,-2}. \quad (75)$$

Notice that the matching equations in (69)–(75) have a recursive structure. The functions $\Pi_{4,0}$, $\Pi_{4,1}$, $\Pi_{4,2}$, $\sigma_{4,-2}$, and $\sigma_{4,0}$ are given explicitly in terms of integrals and sum-integrals. The functions $\Pi_{8,-2}$ and $\sigma_{8,-4}$ are given in terms of integrals of the function $\sigma_{4,-2}(q)$ which has already been determined. The recursive structure of these equations continues at higher order in g . Thus, by using this method, we can solve the variational equation for $\Pi(P)$ order-by-order in g_0^4 and m/T for both hard P and soft P .

In order to calculate the thermodynamic potential, we will need to calculate the sum-integrals and integrals involving some of the functions $\Pi_{m,k}(P)$ and $\sigma_{m,k}(p)$. Using the solutions (69) and (70), we obtain the following sum-integrals:

$$\not\!\!\!\int_P \frac{\Pi_{4,0}(P)}{(P^2)^2} = -\frac{1}{6} \not\!\!\!\int_{PQR} \frac{1}{(P^2)^2 Q^2 R^2 (P+Q+R)^2} + \frac{1}{6}T^2 I_{\text{sun}}(im) \not\!\!\!\int_P \frac{1}{(P^2)^2}, \quad (76)$$

$$\not\!\!\!\int_P \frac{\Pi_{4,1}(P)}{(P^2)^2} = -\frac{1}{2}T I_1 \left[\not\!\!\!\int_{PQ} \frac{1}{(P^2)^2 Q^2 (P+Q)^2} - \left(\not\!\!\!\int_P \frac{1}{(P^2)^2} \right)^2 \right]. \quad (77)$$

Using the solutions (73)–(75), we obtain the following integrals:

$$T \int_{\mathbf{p}} \frac{\sigma_{4,-2}(p)}{(p^2+m^2)^2} = -\frac{1}{6}T^3 \int_{\mathbf{p}} \frac{I_{\text{sun}}(p) - I_{\text{sun}}(im)}{(p^2+m^2)^2}, \quad (78)$$

$$T \int_{\mathbf{p}} \frac{\sigma_{4,0}(p)}{(p^2 + m^2)^2} = \frac{1}{6} T I_1 \not\int_{PQ} \left(\frac{1}{(P^2)^2 Q^2 (P+Q)^2} - \frac{(4/d) \mathbf{p}^2}{(P^2)^3 Q^2 (P+Q)^2} \right), \quad (79)$$

$$T \int_{\mathbf{p}} \frac{[\sigma_{4,-2}(p)]^2}{(p^2 + m^2)^3} = \frac{1}{36} T^5 \int_{\mathbf{p}} \frac{[I_{\text{sun}}(p) - I_{\text{sun}}(im)]^2}{(p^2 + m^2)^3}, \quad (80)$$

$$T \int_{\mathbf{p}} \frac{\sigma_{8,-4}(p)}{(p^2 + m^2)^2} = \frac{1}{36} T^5 \int_{\mathbf{p}} \frac{I_{\text{sun}}(p) - I_{\text{sun}}(im)}{(p^2 + m^2)^2} \frac{d}{dm^2} I_{\text{sun}}(p) + T I_2 \Pi_{8,-2}. \quad (81)$$

C. Gap Equation

Upon inserting our solutions to the variational equation into (57) and expanding it in powers of m/T , we will obtain a thermodynamic potential $\Omega_0(m)$ that depends on a single variational parameter m . This function will contain severe ultraviolet divergences. Our goal will be to construct a thermodynamic potential that is a finite function of m but has the same variational minimum. To accomplish this, we will add to $\Omega_0(m)$ a quantity that includes a factor of $(m^2 - G)^2$, where G is the right side of the gap equation (62). We will therefore need this gap equation in the form of an explicit equation for m .

To make the gap equation (62) explicit, we must expand the sum-integrals on the right side in powers of m/T . We will keep all finite terms through order g^5 and all divergent terms through order g^7 . Our first step will be to expand them in powers of the self-energy functions $\Pi_{n,k}(P)$ and $\sigma_{n,k}(p)$. Each of the sum-integrals in (92) can be expressed as a sum of the corresponding massive sum-integral, sum-integrals over P involving the functions $\Pi_{n,k}(P)$, and integrals over \mathbf{p} involving the functions $\sigma_{n,k}(p)$. The terms that contribute to the gap equation through order g^7 are

$$\not\int_P \frac{1}{P^2 + \Pi(P)} = \mathcal{I}_1 - (g_0 \mu^{-\epsilon})^4 \left[T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q) + \sigma_{4,0}(q)}{(q^2 + m^2)^2} + \not\int_Q \frac{\Pi_{4,0}(Q) + \Pi_{4,1}(Q)}{(Q^2)^2} \right] - (g_0 \mu^{-\epsilon})^8 \int_{\mathbf{q}} \left(\frac{\sigma_{8,-4}(q)}{(q^2 + m^2)^2} - \frac{\sigma_{4,-2}^2(q)}{(q^2 + m^2)^3} \right) + \mathcal{O}(g^6), \quad (82)$$

$$\mathfrak{S}_{\text{screen}} = \mathcal{I}_{\text{screen}} - 3(g_0 \mu^{-\epsilon})^4 \left[T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{k} + \mathbf{q}|) \Big|_{k=im} + T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} \not\int_Q \frac{1}{(Q^2)^2} \right] + \mathcal{O}(g^4), \quad (83)$$

The resulting form of the gap equation, including all term through order $g^7 T^2$, is

$$\begin{aligned}
m^2 = & (g_0\mu^{-\epsilon})^2 \left[\frac{1}{2} \mathcal{I}_1 \right] + (g_0\mu^{-\epsilon})^4 \left[-\frac{1}{6} \mathcal{I}_{\text{screen}} \right] \\
& + (g_0\mu^{-\epsilon})^6 \left\{ -\frac{1}{2} T \int_{\mathbf{p}} \frac{\sigma_{4,-2}(p) + \sigma_{4,0}(p)}{(p^2 + m^2)^2} - \frac{1}{2} \not\int_P \frac{\Pi_{4,0}(P) + \Pi_{4,1}(P)}{(P^2)^2} \right\} \\
& + (g_0\mu^{-\epsilon})^8 \left\{ \frac{1}{2} T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) \Big|_{p=im} + \frac{1}{2} T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} \not\int_P \frac{1}{(P^2)^2} \right\} \\
& + (g_0\mu^{-\epsilon})^{10} \left\{ -\frac{1}{2} T \int_{\mathbf{q}} \left(\frac{\sigma_{8,-4}(q)}{(q^2 + m^2)^2} - \frac{\sigma_{4,-2}^2(q)}{(q^2 + m^2)^3} \right) \right\} + \mathcal{O}(g^8 T^2). \tag{84}
\end{aligned}$$

The m/T expansions of the massive sum-integrals \mathcal{I}_1 and $\mathcal{I}_{\text{screen}}$ are given in (D.6) and (D.9). The only momentum integrals involving self-energy functions that contribute to the finite terms through order $g^5 T^2$ or to divergent terms through order $g^7 T^2$ are

$$T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} = \frac{T^2}{(4\pi)^4} \frac{1 - \log 2}{6} \frac{1}{\hat{m}}, \tag{85}$$

$$T \int_{\mathbf{q}} \frac{\sigma_{4,0}(q)}{(q^2 + m^2)^2} = \frac{T^2}{(4\pi)^4} \left[-\frac{1}{48\epsilon} \right] \hat{m}. \tag{86}$$

The sum-integrals involving self-energy functions are

$$\not\int_P \frac{\Pi_{4,0}(P)}{(P^2)^2} = \frac{T^2}{48(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left[\frac{1}{\epsilon^2} + \left(-8 \log \hat{m} - \frac{17}{6} - 16 \log 2 - 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} \right], \tag{87}$$

$$\not\int_P \frac{\Pi_{4,1}(P)}{(P^2)^2} = -\frac{T^2}{8(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left[\frac{1}{\epsilon^2} + (-2 \log \hat{m} + 1 + 4\gamma) \frac{1}{\epsilon} \right] \hat{m}. \tag{88}$$

Keeping terms of order ϵ through order g^3 , finite terms through order g^5 , and divergent terms through order g^7 , the gap equation (84) becomes

$$\begin{aligned}
\hat{m}^2 = & (\alpha_0 \mu^{-2\epsilon}) \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left\{ \frac{1}{6} \left[1 + \left(2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon \right] - [1 + (-2 \log \hat{m} + 2) \epsilon] \hat{m} \right. \\
& \left. - \frac{1}{2} \left[\frac{1}{\epsilon} + 2\gamma \right] \hat{m}^2 \right\} \\
& + (\alpha_0 \mu^{-2\epsilon})^2 \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left\{ -\frac{1}{6} \left[\frac{1}{\epsilon} + (-4 \log \hat{m} + 6 - 8 \log 2) \right] \right. \\
& \left. + \left[\frac{1}{\epsilon} + (-2 \log \hat{m} + 2 + 2\gamma) \right] \hat{m} \right. \\
& \left. + \frac{1}{4} \left[\frac{1}{\epsilon^2} + \left(\frac{5}{6} + 4\gamma \right) \frac{1}{\epsilon} \right] \hat{m}^2 \right\} \\
& + (\alpha_0 \mu^{-2\epsilon})^3 \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left\{ -\frac{1 - \log 2}{3} \frac{1}{\hat{m}} \right. \\
& \left. - \frac{1}{24} \left[\frac{1}{\epsilon^2} + \left(-8 \log \hat{m} - \frac{17}{6} - 16 \log 2 - 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left[\frac{1}{\epsilon^2} + \left(-2 \log \hat{m} + \frac{7}{6} + 4\gamma \right) \frac{1}{\epsilon} \right] \hat{m} \Big\} \\
& + (\alpha_0 \mu^{-2\epsilon})^4 \left(\frac{\hat{\mu}}{2} \right)^{8\epsilon} \left\{ \frac{1 - \log 2}{3\epsilon} \frac{1}{\hat{m}} \right\}, \tag{89}
\end{aligned}$$

D. Thermodynamic Potential

The 3-loop Φ -derivable approximation to the thermodynamic potential $\Omega_0[D]$ is given in (57) as a functional of the self-energy $\Pi(P)$. The sum-integrals can be expanded in powers of m/T using the methods of Appendix A. The function $\Pi(P)$ can then be eliminated in favor of the mass variable m by using the solution to the variational equation in subsection V.B. This reduces the thermodynamic potential to a function $\Omega_0(m)$ of a single variational parameter m . This function contains many ultraviolet divergences. Some of them are eliminated when $\Omega_0(m)$ is evaluated at the solution of the gap equation. Some can be eliminated by renormalization of the coupling constant. There are other ultraviolet divergences that cannot be canceled and represent unavoidable ambiguities in the Φ -derivable approximation. We wish to construct a thermodynamic potential that is finite to as high an order in g as possible.

Our first step in constructing a finite thermodynamic potential $\Omega(m)$ will be to cancel the most severe divergences that are eliminated by evaluating $\Omega_0(m)$ at the solution to the gap equation. These are the divergences proportional to m^4/ϵ and $g_0^2 m^4/\epsilon^2$ that were already encountered in the 2-loop Φ -derivable approximation. They can be canceled without changing the free energy by adding the term

$$- \frac{1}{2(g_0 \mu^{-\epsilon})^2} \left[\Pi(0) - \frac{1}{2}(g_0 \mu^{-\epsilon})^2 \not\!\!\!\int_Q \frac{1}{Q^2 + \Pi(Q)} + \frac{1}{6}(g_0 \mu^{-\epsilon})^4 \mathfrak{S}_{\text{sun}}(0) \right]^2. \tag{90}$$

Since this is proportional to the square of the gap equation evaluated at $P = 0$, it vanishes at the solution to the gap equation. Using (64) to eliminate $\Pi(0)$ in favor of m , this term can be written

$$- \frac{1}{2(g_0 \mu^{-\epsilon})^2} \left[m^2 - \frac{1}{2}(g_0 \mu^{-\epsilon})^2 \not\!\!\!\int_Q \frac{1}{Q^2 + \Pi(Q)} + \frac{1}{6}(g_0 \mu^{-\epsilon})^4 \mathfrak{S}_{\text{screen}} \right]^2. \tag{91}$$

The thermodynamic potential $\Omega_1(m)$ defined by adding (91) to (57) is

$$\begin{aligned}
\mu^{2\epsilon}\Omega_1(m) = & -\frac{m^4}{2(g_0\mu^{-\epsilon})^2} + \frac{1}{2}\oint_P \left(\log(P^2 + \Pi(P)) - \frac{\Pi(P) - m^2}{P^2 + \Pi(P)} \right) \\
& + (g_0\mu^{-\epsilon})^2 \left[-\frac{1}{6}m^2\mathfrak{S}_{\text{screen}} \right] \\
& + (g_0\mu^{-\epsilon})^4 \left[-\frac{1}{48}\mathfrak{S}_{\text{ball}} + \frac{1}{12}\mathfrak{S}_{\text{screen}}\oint_P \frac{1}{P^2 + \Pi(P)} \right] \\
& + (g_0\mu^{-\epsilon})^6 \left[-\frac{1}{72}\mathfrak{S}_{\text{screen}}^2 \right]. \tag{92}
\end{aligned}$$

We want to expand the thermodynamic potential (92) in powers of m/T , keeping all finite terms through order g^5 and all divergent terms through order g^7 . Our first step will be to expand in powers of the self-energy functions $\Pi_{n,k}(P)$ and $\sigma_{n,k}(p)$. Each of the sum-integrals in (92) can be expressed as a sum of the corresponding massive sum-integral, sum-integrals over P involving the functions $\Pi_{n,k}(P)$, and integrals over \mathbf{p} involving the functions $\sigma_{n,k}(p)$. For the tadpole sum-integral and for $\mathfrak{S}_{\text{screen}}$, these expressions are given in (82) and (83). For the other two sum-integrals in (92), the terms that contribute to the thermodynamic potential through order g^7 are

$$\begin{aligned}
& \oint_P \left(\log(P^2 + \Pi(P)) - \frac{\Pi(P) - m^2}{P^2 + \Pi(P)} \right) \\
& = -\mathcal{I}'_0 + \frac{1}{2}(g_0\mu^{-\epsilon})^8 T \int_{\mathbf{q}} \frac{\sigma_{4,-2}^2(q)}{(q^2 + m^2)^2} + \mathcal{O}(g^8 T^4), \tag{93}
\end{aligned}$$

$$\mathfrak{S}_{\text{ball}} = \mathcal{I}_{\text{ball}} - 4(g_0\mu^{-\epsilon})^4 T^3 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q) I_{\text{sun}}(q)}{(q^2 + m^2)^2} + \mathcal{O}(g^4 T^4). \tag{94}$$

The resulting expression for the thermodynamic potential, including all terms through order g^7 , is

$$\begin{aligned}
\mu^{2\epsilon}\Omega_1(m) = & -\frac{m^4}{2(g_0\mu^{-\epsilon})^2} + \frac{1}{2}(-\mathcal{I}'_0) + (g_0\mu^{-\epsilon})^2 \left[-\frac{1}{6}m^2\mathcal{I}_{\text{screen}} \right] \\
& + (g_0\mu^{-\epsilon})^4 \left[-\frac{1}{48}\mathcal{I}_{\text{ball}} + \frac{1}{12}\mathcal{I}_{\text{screen}}\mathcal{I}_1 \right] \\
& + (g_0\mu^{-\epsilon})^6 \left[-\frac{1}{72}\mathcal{I}_{\text{screen}}^2 + \frac{1}{2}m^2 T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{k} + \mathbf{q}|) \Big|_{k=im} \right. \\
& \quad \left. + \frac{1}{2}m^2 T \oint_P \frac{1}{(P^2)^2} \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + (g_0 \mu^{-\epsilon})^8 \left[-\frac{1}{4} \mathcal{I}_1 T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{k} + \mathbf{q}|) \Big|_{k=im} \right. \\
& \quad - \frac{1}{12} \left(\mathcal{I}_{\text{screen}} + 3 \mathcal{I}_1 \cancel{\int_Q} \frac{1}{(Q^2)^2} \right) T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} \\
& \quad \left. + \frac{1}{4} T \int_{\mathbf{q}} \frac{\sigma_{4,-2}^2(q)}{(q^2 + m^2)^2} + \frac{1}{12} T^3 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q) I_{\text{sun}}(q)}{(q^2 + m^2)^2} \right] + \mathcal{O}(g^8 T^4). \quad (95)
\end{aligned}$$

The next step is to express the coefficient of each power of g_0 as an expansion in powers of m/T . The expansions of the sum-integrals \mathcal{I}'_0 , \mathcal{I}_1 , $\mathcal{I}_{\text{screen}}$, and $\mathcal{I}_{\text{ball}}$ are given in Appendix D. The integrals involving the self-energy function $\sigma_{4,-2}(q)$ are given explicitly in (78)-(81). The only integrals that contribute to the finite terms in (95) through order g^5 or to divergent terms through order g^7 are (85) and

$$T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q) I_{\text{sun}}(q)}{(q^2 + m^2)^2} = \frac{T^2}{(4\pi)^6} \frac{1 - \log 2}{24\epsilon} \frac{1}{\hat{m}}. \quad (96)$$

The complete m/T expansion of the thermodynamic potential, including all finite terms through order g^5 and all divergent terms through order g^7 , is

$$\begin{aligned}
& \frac{1}{15} \left(\frac{\mu^{2\epsilon} \Omega_1(m)}{\mathcal{F}_{\text{ideal}}} - 1 \right) = \frac{3}{\alpha_0 \mu^{-2\epsilon}} \hat{m}^4 \\
& + \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left[-\hat{m}^2 + 4\hat{m}^3 + \frac{3}{2} \left(\frac{1}{\epsilon} + 2\gamma \right) \hat{m}^4 \right] \\
& + (\alpha_0 \mu^{-2\epsilon}) \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left\{ \left(\frac{1}{\epsilon} - 4 \log \hat{m} + 6 - 8 \log 2 \right) \hat{m}^2 \right. \\
& \quad - 6 \left(\frac{1}{\epsilon} - 2 \log \hat{m} + 2 + 2\gamma \right) \hat{m}^3 \\
& \quad \left. - \frac{3}{2} \left[\frac{1}{\epsilon^2} + \left(\frac{5}{6} + 4\gamma \right) \frac{1}{\epsilon} \right] \hat{m}^4 \right\} \\
& + (\alpha_0 \mu^{-2\epsilon})^2 \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left\{ -\frac{1}{12} \left[\frac{1}{\epsilon} - 8 \log \hat{m} + \frac{149}{15} - 16 \log 2 - 4 \frac{\zeta'(-1)}{\zeta(-1)} + 2 \frac{\zeta'(-3)}{\zeta(-3)} \right] \right. \\
& \quad + \left[\frac{1}{\epsilon} - 2 \log \hat{m} + 4 - 4 \log 2 + 2\gamma + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] \hat{m} \\
& \quad + \frac{1}{2} \left[\frac{1}{\epsilon^2} + (-4 \log \hat{m} - 6 - 8 \log 2 + 2\gamma) \frac{1}{\epsilon} \right] \hat{m}^2 \\
& \quad \left. - 3 \left[\frac{1}{\epsilon^2} + (-2 \log \hat{m} + 2 + 4\gamma) \frac{1}{\epsilon} \right] \hat{m}^3 \right\} \\
& + (\alpha_0 \mu^{-2\epsilon})^3 \left(\frac{\hat{\mu}}{2} \right)^{8\epsilon} \left\{ \frac{1}{12} \left[\frac{1}{\epsilon^2} + (-8 \log \hat{m} + 12 - 16 \log 2) \frac{1}{\epsilon} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{1}{\epsilon^2} + (-6 \log \hat{m} + 10 - 10 \log 2 + 2\gamma) \frac{1}{\epsilon} \right] \hat{m} \Big\} \\
& + (\alpha_0 \mu^{-2\epsilon})^4 \left\{ \frac{1 - \log 2}{3\epsilon} \frac{1}{\hat{m}} \right\}.
\end{aligned} \tag{97}$$

This expression has ultraviolet divergences of order g^4 and higher. We have two ways to eliminate these divergences. One is to use (3) to eliminate the bare coupling constant α_0 in favor of the renormalized coupling constant $\alpha(\mu)$. The other is to add to the thermodynamic potential $\Omega(m)$ something proportional to the square of the gap equation. In particular, we can add a term such as (91), with a coefficient proportional to any power of $(g_0 \mu^{-\epsilon})^2$. We can define a thermodynamic potential $\Omega_2(m)$ in which the divergences are postponed to order g^6 as follows:

$$\frac{\mu^{2\epsilon} \Omega_2(m)}{15 \mathcal{F}_{\text{ideal}}} = \frac{\mu^{2\epsilon} \Omega_1(m)}{15 \mathcal{F}_{\text{ideal}}} + \frac{3}{\epsilon} (\hat{m}^2 - \hat{G})^2 \left(\frac{\mu}{\Lambda} \right)^{2\epsilon}, \tag{98}$$

where \hat{G} is the right side of the gap equation (89). The factor of $\mu^{2\epsilon}$ in the term that is added guarantees that $\Omega_2(m)$ will be independent of μ . The price that must be paid for the independence of μ is the introduction of another arbitrary scale Λ . After making the substitution (3) and truncating after terms of 5th order in \hat{m} or g , we obtain a finite thermodynamic potential

$$\begin{aligned}
\frac{1}{15} \left(\frac{\Omega(m)}{\mathcal{F}_{\text{ideal}}} - 1 \right) &= \frac{3\hat{m}^4}{\alpha} + \left[-\hat{m}^2 + 4\hat{m}^3 + 3(3L - 2\ell + \gamma) \hat{m}^4 \right] \\
&+ \alpha \left[2 \left(\ell - 2 \log \hat{m} + 2 - 4 \log 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) \hat{m}^2 - 12(\ell + \gamma) \hat{m}^3 \right] \\
&+ \alpha^2 \left[-\frac{1}{6} \left(\ell - 4 \log \hat{m} + \frac{89}{30} - 8 \log 2 - 4 \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-3)}{\zeta(-3)} \right) \right. \\
&\quad \left. + 2(\ell - 2 \log 2 + \gamma) \hat{m} \right],
\end{aligned} \tag{99}$$

where $\ell = \log(\Lambda/4\pi T)$. This is our final result for the 3-loop Φ -derivable thermodynamic potential. The divergent terms of order g^6 that were eliminated by the truncation are

$$\begin{aligned}
& \frac{3}{2} \left[\frac{1}{\epsilon^2} + (4L - 4\ell + 2) \frac{1}{\epsilon} \right] \alpha \hat{m}^4 \\
& + \left[\frac{1}{\epsilon^2} + \left(6L + 2\ell - 12 \log \hat{m} + 11 - 24 \log 2 - 4 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} \right] \alpha^2 \hat{m}^2
\end{aligned}$$

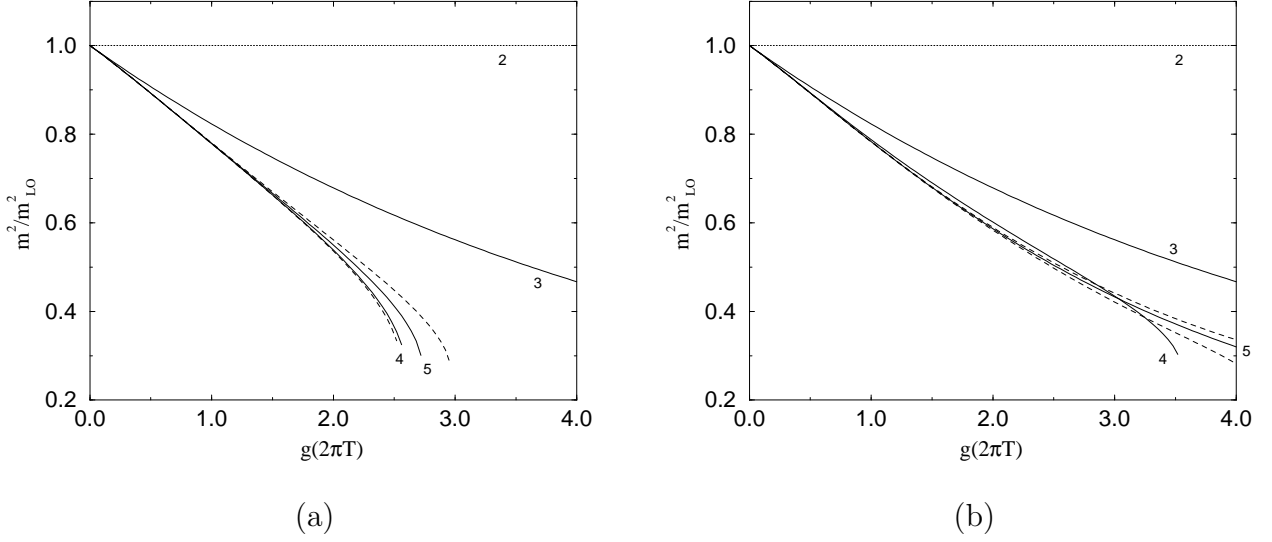


FIG. 4. Φ -derivable approximation to the screening mass divided by the leading order solution $m_{LO}^2 = g^2(2\pi T)T^2/24$ as a function of $g(2\pi T)$ for (a) $\Lambda = 2\pi T$ and (b) $\Lambda = m$. Solid lines are solutions to the gap equation truncated at orders g^n , $n = 2, 3, 4, 5$ with $\mu = 2\pi T$. Dashed lines are solutions to the gap equation truncated at order g^5 for $\mu = \pi T$ and $\mu = 4\pi T$.

$$-\frac{1}{12} \left[\frac{1}{\epsilon^2} + \left(8L + 2\ell - 24 \log \hat{m} + \frac{109}{5} - 48 \log 2 - 20 \frac{\zeta'(-1)}{\zeta(-1)} + 6 \frac{\zeta'(-3)}{\zeta(-3)} \right) \frac{1}{\epsilon} \right] \alpha^3. \quad (100)$$

Note that the double poles in ϵ do not cancel if we use the leading order gap equation $\hat{m}^2 = \alpha/6$. Thus they cannot be eliminated by adding another term to the thermodynamic potential proportional to $(\hat{m}^2 - \hat{G})^2$. These ultraviolet divergences arise from having truncated the interaction functional Φ at 3 loops. To eliminate them, one would have to include 4-loop terms in Φ .

In the case of the 2-loop Φ -derivable approximation, we were able to eliminate the ultraviolet divergences to all orders by replacing the true coupling constant α by a new coupling constant $\bar{\alpha}$ whose expansion in α had divergent coefficients but which still had a well-defined beta function. If we demand that the resulting beta function has the correct one-loop coefficient, the new coupling constant must have the form

$$\bar{\alpha} = \alpha + \frac{A}{\epsilon} \alpha^3 + \dots \quad (101)$$

where A is a constant. By an appropriate choice of this constant, one might be able to eliminate one of the terms of order \bar{g}^6 with a single pole in ϵ . But since we cannot eliminate the double poles, there is nothing to be gained by introducing such a coupling constant.

E. Screening Mass

The gap equation obtained by varying our finite thermodynamic potential (99) with respect to m^2 while holding μ and Λ fixed is

$$\begin{aligned}\hat{m}^2 = & \alpha \left[\frac{1}{6} - \hat{m} - (3L - 2\ell + \gamma) \hat{m}^2 \right] \\ & + \alpha^2 \left[-\frac{1}{3} \left(\ell - 2 \log \hat{m} + 1 - 4 \log 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) + 3(\ell + \gamma) \hat{m} \right] \\ & + \alpha^3 \left[-\frac{1}{18} \frac{1}{\hat{m}^2} - \frac{1}{6} (\ell - 2 \log 2 + \gamma) \frac{1}{\hat{m}} \right].\end{aligned}\tag{102}$$

The solution to this equation is the 3-loop Φ -derivable approximation to the screening mass m_s . If we solve the equation iteratively in powers of $(\alpha/6)^{1/2}$, it agrees with the weak-coupling expansion (14) for the screening mass. The screening mass depends on the arbitrary scale Λ through the logarithms ℓ . It also depends on the renormalization scale μ through $\alpha(\mu)$ and L . However since it agrees with the weak-coupling expansion through order g^5 , all dependence on these arbitrary scales must be of order g^6 or higher.

We can obtain a series of successive approximations to the screening mass by truncating the gap equation (102) after terms of n^{th} order in g and \hat{m} . In Fig. 4a, we show the solutions for m^2 divided by the leading order result $m_{\text{LO}}^2 = g^2(2\pi T)T^2/24$ for $\mu = \Lambda = 2\pi T$ as a function of $g(2\pi T)$. The solid lines correspond to truncations after orders $n = 2, 3, 4, 5$. For $n = 4$ and 5 , there is no solution beyond a critical value of the coupling constant: $g = 2.591$ for $n = 4$ and $g = 2.768$ for $n = 5$. For larger values of g , the thermodynamic potential has a runaway minimum $m \rightarrow 0$ that comes from the $\alpha^2 \log \hat{m}$ term in (99). A similar phenomenon occurs in screened perturbation theory at three loops [21]: if the screening mass is used as the variational parameter, the solution to the gap equation terminates at nearly the same critical value of g . For g below the critical value, the screening mass seems

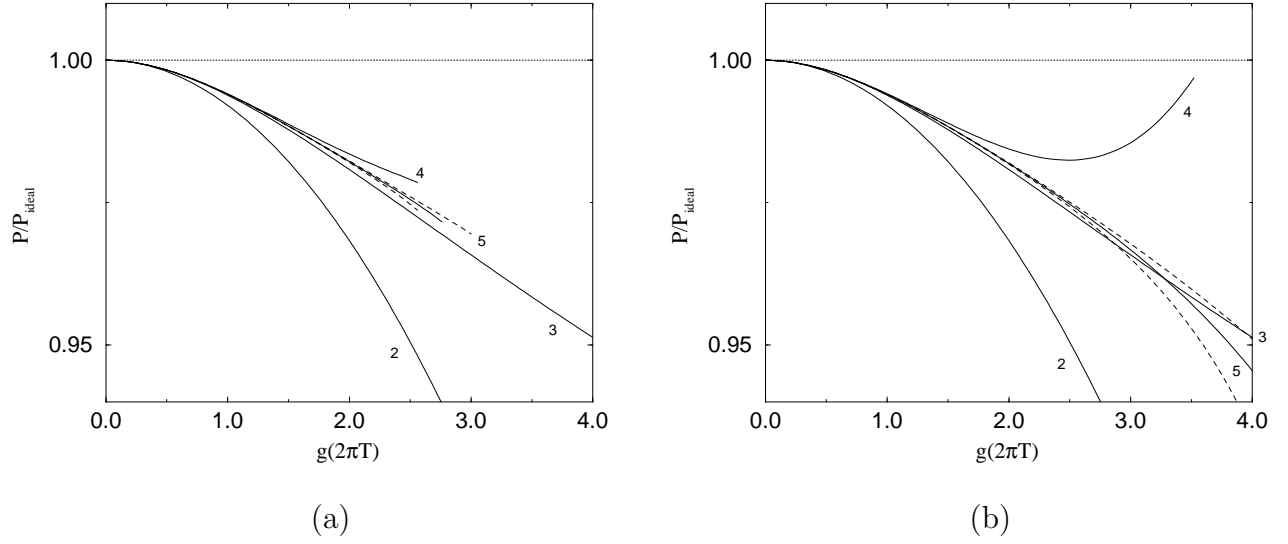


FIG. 5. Φ -derivable approximation to the pressure divided by that of the ideal gas as a function of $g(2\pi T)$ for (a) $\Lambda = 2\pi T$ and (b) $\Lambda = m$. Solid lines correspond to the truncation of the thermodynamic potential at order g^n , $n = 2, 3, 4, 5$ for $\mu = 2\pi T$. Dashed lines correspond to the truncation at order g^5 for $\mu = \pi T$ and $\mu = 4\pi T$.

to be converging nicely. It is also fairly insensitive to the renormalization scale, as shown in Fig. 4a. The solution to the gap equation truncated at 5th order is shown as dashed lines for the two choices $\mu = \pi T$ and $\mu = 4\pi T$.

The simplest way to get a solution for the screening mass that extends beyond the critical value of g is to choose the arbitrary scale Λ to be proportional to m . This changes the gap equation to

$$\begin{aligned} \hat{m}^2 = & \alpha \left[\frac{1}{6} - \hat{m} - \left(3L - 2\ell - \frac{1}{2} + \gamma \right) \hat{m}^2 \right] \\ & + \alpha^2 \left[-\frac{1}{3} \left(\ell - 2 \log \hat{m} + \frac{3}{2} - 4 \log 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) + 3 \left(\ell + \gamma + \frac{1}{3} \right) \hat{m} \right] \\ & + \alpha^3 \left[-\frac{1}{24} \frac{1}{\hat{m}^2} - \frac{1}{6} (\ell + 1 - 2 \log 2 + \gamma) \frac{1}{\hat{m}} \right]. \end{aligned} \quad (103)$$

When truncated at 2nd and 3rd order in g and \hat{m} , the gap equations (102) and (103) are identical. The solution to the gap equation (103) truncated at n^{th} order and with $\mu = 2\pi T$ and $\Lambda = m$ are shown as solid lines in Fig. 4b. The solution for $n = 4$ terminates at

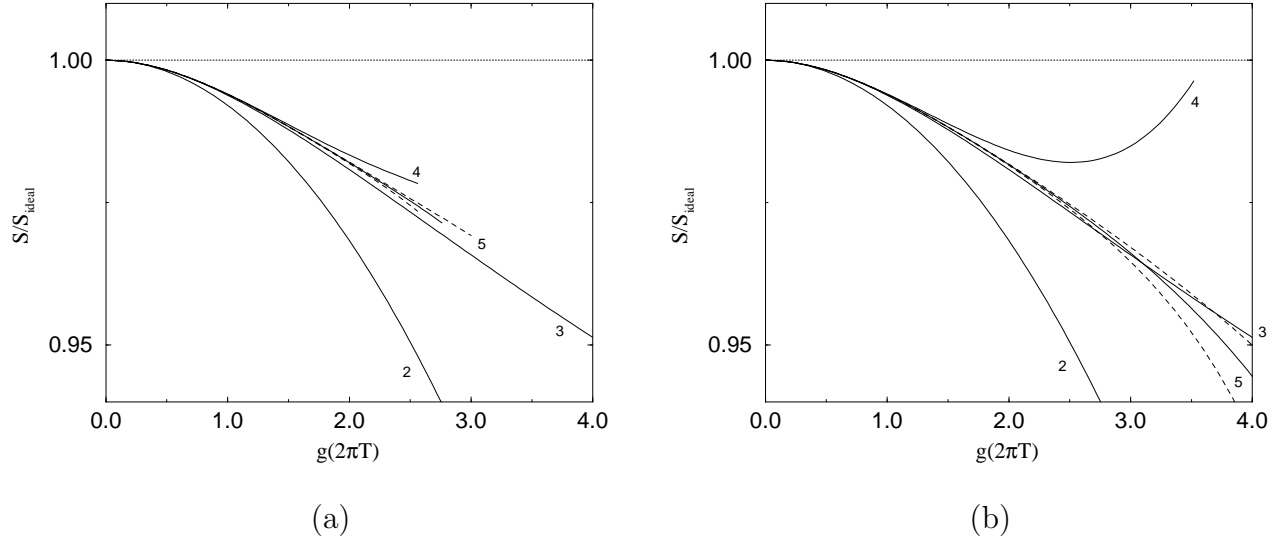


FIG. 6. Φ -derivable approximation to the entropy divided by that of the ideal gas as a function of $g(2\pi T)$ for (a) $\Lambda = 2\pi T$ and (b) $\Lambda = m$. Solid lines correspond to the truncation of the entropy at order g^n , $n = 2, 3, 4, 5$ for $\mu = 2\pi T$. Dashed lines correspond to the truncation at order g^5 for $\mu = \pi T$ and $\mu = 4\pi T$.

$g = 3.551$ for $\mu = 2\pi T$. The solution for $n = 5$ continues well beyond $g(2\pi T) = 4$. The dashed lines are the 5th order truncation with $\Lambda = m$ for two choices of the renormalization scale: $\mu = \pi T$ and $\mu = 4\pi T$.

F. Thermodynamic Functions

The pressure, given by (8), is obtained by inserting the solution to the gap equation (102) into the thermodynamic potential (99). The pressure truncated at n^{th} order in g and \hat{m} is obtained by using the n^{th} order truncations of (102) and (99). In Fig. 5a, we show the n^{th} order truncations of the pressure divided by that of the ideal gas for $\Lambda = 2\pi T$ and $\mu = 2\pi T$. We show also the change in the 5th order truncation from varying the renormalization scale μ . Since the solutions to the 4th order and 5th order gap equations cannot be continued beyond the critical values $g(2\pi T) = 2.591$ and 2.768 respectively, the corresponding lines for the pressure cannot be extended past these values. The variations of the pressure are much

smaller than one might expect from the variations of the screening mass in Fig. 4a. We see an obvious improvement in the convergence compared to the weak-coupling expansion. At 5th order, and up to the critical values of $g(2\pi T)$ starting around 2.5, the variations from the renormalization scale are much smaller than those of the weak-coupling expansion. Fig. 5b shows the pressure truncated at the same orders but with the scale $\Lambda = m$. The pressure is obtained by inserting the solution to the gap equation (103) into the thermodynamic potential (99). The line for $n = 5$ continues well beyond $g(2\pi T) = 4$, but for values of $g(2\pi T)$ around 2.5 the convergence between $n = 3, 4$ and 5 is not as good as in the previous case of $\Lambda = 2\pi T$. Up to $g(2\pi T) = 4$, the changes in the 5th order truncation from varying the renormalization scale μ appears to be much smaller than in the weak-coupling expansion.

The diagrammatic entropy is given by Eq. (9), with μ and $\alpha(\mu)$ held fixed. We have to consider two cases giving different expressions of the entropy, the case with $\Lambda = c(2\pi T)$ and the case with $\Lambda = c m$. In the first case, upon differentiating the thermodynamic potential (99) with respect to T , the 5th order expression for the entropy is

$$\begin{aligned} \frac{1}{15} \left(\frac{\mathcal{S}}{\mathcal{S}_{\text{ideal}}} - 1 \right) &= \frac{1}{4} \left[-2\hat{m}^2 + 4\hat{m}^3 - 9\hat{m}^4 \right] \\ &+ \alpha \left[\left(\ell - 2 \log \hat{m} + 3 - 4 \log 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) \hat{m}^2 - 3(\ell + \gamma) \hat{m}^3 \right] \\ &+ \alpha^2 \left[-\frac{1}{6} \left(\ell - 4 \log \hat{m} + \frac{119}{30} - 8 \log 2 - 4 \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-3)}{\zeta(-3)} \right) \right. \\ &\quad \left. + \frac{3}{2} (\ell - 2 \log 2 + \gamma) \hat{m} \right], \end{aligned} \quad (104)$$

where $\mathcal{S}_{\text{ideal}} = (2\pi^2/45)T^3$, and \hat{m} is the solution of the gap equation (102). Fig. 6a displays the good convergence of the n^{th} order truncation of the entropy for $\mu = 2\pi T$ and $\Lambda = 2\pi T$, and the small variations from the renormalization scale μ . The 5th order expression for the entropy for the choice $\Lambda = c m$ is given by

$$\begin{aligned} \frac{1}{15} \left(\frac{\mathcal{S}}{\mathcal{S}_{\text{ideal}}} - 1 \right) &= \frac{1}{4} \left[-2\hat{m}^2 + 4\hat{m}^3 - 3\hat{m}^4 \right] \\ &+ \alpha \left[\left(\ell - 2 \log \hat{m} + \frac{5}{2} - 4 \log 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) \hat{m}^2 - 3(\ell - 1 + \gamma) \hat{m}^3 \right] \\ &+ \alpha^2 \left[-\frac{1}{6} \left(\ell - 4 \log \hat{m} + \frac{223}{60} - 8 \log 2 - 4 \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-3)}{\zeta(-3)} \right) \right. \end{aligned}$$

$$+\frac{3}{2}\left(\ell-\frac{1}{3}-2\log 2+\gamma\right)\hat{m}\Big], \quad (105)$$

with \hat{m} the solution to the gap equation (103) and $\ell = \log(m/2\pi T)$. Fig. 6b shows the n^{th} order truncations of the entropy for $\Lambda = m$. We conclude again, as in the case of the pressure, that around the critical values of $g(2\pi T)$ the convergence is not as good as in the case of $\Lambda = 2\pi T$. However the changes of the 5th order truncation with respect to the variations of the renormalization scale μ are still relatively small.

VI. CONCLUSIONS

We have developed a systematic method for solving Φ -derivable approximations for the massless ϕ^4 field theory. The method involves expanding sum-integrals systematically in powers of m/T , where m is the screening mass. When expanded in powers of m/T , the variational equations for the self-energy $\Pi(P)$ have a recursive structure that allows them to be solved in terms of the variable m . This reduces the problem to one that involves the single variational parameter m . Upon truncating the expansion in m/T , we obtain a gap equation for m that could easily be solved numerically were it not for the presence of ultraviolet divergences. Some of the ultraviolet divergences vanish at the variational point, and can be eliminated by a redefinition of the thermodynamic potential. However there are other ultraviolet divergences that do not vanish at the variational point. They are generated by the truncation error in defining the interaction functional $\Phi[\Pi]$. They limit the accuracy to which the Φ -derivable approximation can be solved.

In the case of the 2-loop Φ -derivable approximation, the unavoidable ultraviolet divergences appear at order g^4 . They can be eliminated to all orders by a trick that involves introducing a new coupling constant \bar{g} that runs with the wrong beta function. However this trick does not generalize to higher orders in the loop expansion. If one expresses the thermodynamic potential in terms of the true coupling constant g , the best one can do in constructing a finite thermodynamic potential is to truncate after terms of third order in

m/T .

In the 3-loop Φ -derivable approximation, the unavoidable ultraviolet divergences appear at order g^6 . We constructed a finite thermodynamic potential by adding terms proportional to the square of the gap equation to eliminate the ultraviolet divergences that vanish at the variational point and then truncating after terms of 5th order in g and m/T . Truncations of the finite thermodynamic potential at n^{th} order in g and m/T , where $n = 2, 3, 4, 5$, define a sequence of successive variational approximations to the thermodynamics. The truncation after terms of 3rd order in g and m/T reproduces the finite thermodynamic potential defined by the 2-loop Φ -derivable approximation. Our solution of the 3-loop Φ -derivable approximation allows the 4th order and 5th order approximations to be constructed. The 6th order and 7th order approximations could be constructed by applying our method to the 4-loop Φ -derivable approximation. The only obstacle is evaluating the scalar sum-integrals that would arise. The most difficult of the sum-integrals are the same ones that would need to be evaluated to extend the weak-coupling expansion to order g^6 .

We calculated numerically the thermodynamic functions defined by the finite thermodynamic potential truncated at n^{th} order in g and m/T for $n = 2, 3, 4$, and 5. One problem is that in the cases $n = 4$ and 5, the solution to the gap equation for m cannot be extended beyond a critical value of g that is near $g = 2.5$ if we choose the renormalization scale $\mu = 2\pi T$. For values of g below the critical value, the successive approximations seem to be converging and are rather insensitive to variations in the renormalization scale.

Our construction of a finite thermodynamic potential involved a prescription for removing from the bare thermodynamic potential Ω_0 ultraviolet divergences that vanish at the variational point. Our prescription was to add the two terms in (91) and (98) that are both proportional to the square of the gap equation. We had no compelling motivation for this prescription beyond its success in the 2-loop Φ -derivable approximation. This prescription introduces an ambiguity into the finite thermodynamic potential (99). We can feel free to add additional terms proportional to $(\hat{m}^2 - \hat{G})^2$, where \hat{G} is the right side of the finite gap

equation given in (102) or (103). If we avoid large coefficients for these additional terms, they do not have a large effect numerically for g less than about 2.5. However the ambiguity emphasizes that our prescription was rather ad hoc. It would be desirable to have a much more compelling prescription for eliminating the ultraviolet divergences that vanish at the variational point.

Our strategy for solving Φ -derivable approximations should also be applicable to more complicated field theories, such as gauge theories. In the case of QCD , Blaizot, Iancu and Rebhan [14] have solved the 2-loop Φ -derivable equations for the self-energy tensor $\Pi^{\mu\nu}$ to leading order in g_s in terms of a Debye screening mass parameter m_D . They did not determine m_D variationally, but simply used the asymptotic weak-coupling expression $m_D = g_s T$. The resulting approximation reproduces the thermodynamic functions through order g_s^2 . They called it a “leading order” (LO) *HTL* approximation. The application of our method to gauge theories would involve expanding sum-integrals systematically in powers of g_s and m_D/T . If applied to the 2-loop Φ -derivable approximation for QCD , it should reproduce the leading order solution for $\Pi^{\mu\nu}(P)$ in Ref. [14], but it should give a gap equation for m_D that allows it to be determined variationally. By truncating after terms of 3rd order in g_s and m_D/T , one should be able to improve the accuracy of the thermodynamic functions to order g_s^3 . At higher orders in g_s and m_D/T , the 2-loop Φ -derivable approximation will be plagued by ultraviolet divergences associated with the truncation of Φ as well as by gauge dependence associated with the use of the gauge-dependent gluon propagator as a variational function. To improve the accuracy while avoiding these problems, one would have to apply our method to the 3-loop Φ -derivable approximation to QCD . This would allow the accuracy to be improved to orders g_s^4 and g_s^5 . Any further improvement in the accuracy would be very difficult. One problem is that there are new sum-integrals that enter at order g_s^6 that have never been evaluated. A more serious problem is that the 3-loop free energy is sensitive to the nonperturbative momentum scale $g_s^2 T$ associated with magnetic screening. This will probably cause our method for reducing the Φ -derivable approximation

to a single-parameter variational problem to breakdown.

Blaizot, Iancu, and Rebhan [14] have introduced an improvement on their LO *HTL* approximation that reproduces the thermodynamic functions to order g_s^3 . They call it a “next-to-leading order” (NLO) *HTL* approximation. It reduces essentially to the substitution

$$m_D^2 \longrightarrow \frac{g_s^2 T^2}{1 + 3g_s/\pi} \quad (106)$$

in their LO *HTL* approximation. The simpler prescription $m_D^2 \rightarrow g_s^2 T^2 (1 - 3g_s/\pi)$ gives the same parametric improvement in the accuracy, but is a disaster phenomenologically, giving a negative screening mass unless $\alpha_s < 0.087$. The ad hoc prescription (106) gives phenomenologically acceptable results that are not very different from the LO *HTL* approximation of Ref. [14]. We argue that such a prescription is unlikely to arise in any systematically improvable approximation to the thermodynamics. Our solution to the Φ -derivable approximation to the massless ϕ^4 field theory suggests an alternative prescription. The truncation of a gap equation after terms that are 3rd order in g_s and m_D/T as in (56) would give

$$m_D \longrightarrow g_s T \left[\left(1 + \frac{9}{4\pi^2} g_s^2 \right)^{1/2} - \frac{3}{2\pi} g_s \right]. \quad (107)$$

In summary, we have developed a new method for solving the 3-loop Φ -derivable approximation for the massless ϕ^4 field theory. This method can be used to construct a sequence of systematically improvable approximations to the thermodynamic functions. This sequence of approximations seems to be stable and converging for $g(2\pi T)$ less than 2.5, and also insensitive to the choice of renormalization scale. If this method can be adapted to gauge theories, it may provide a practical solution to the longstanding problem of large radiative corrections to the thermodynamic functions.

ACKNOWLEDGMENTS

This work was supported in part by the U. S. Department of Energy under grant DE-FG02-91-ER40690. We thank J. O. Andersen and M. Strickland for valuable discussions.

APPENDIX A: EXPANSIONS IN M/T

In this Appendix, we carry out the expansions of sum-integrals in m/T . We first illustrate the method by applying it to one-loop sum-integrals for a particle of mass m . We then apply it to sum-integrals with a momentum dependent self-energy $\Pi(P)$.

a. Simple examples

To illustrate the m/T expansion in the simplest context, we carry it out for the one-loop sum-integrals that appear in the 2-loop Φ -derivable approximation. The sum-integrals over $P = (2\pi nT, \mathbf{p})$ in (21) and (22) include sums over all Matsubara frequencies n and integrals over all momenta \mathbf{p} . There are two important momentum scales: the *hard* scale $2\pi T$ and the *soft* scale m . The hard region for the momentum $P = (2\pi nT, \mathbf{p})$ includes $n \neq 0$ for all \mathbf{p} and also $n = 0$ with \mathbf{p} of order T . The soft region is $n = 0$ and \mathbf{p} of order m . We will evaluate the contributions to the sum-integrals from these two momentum regions separately.

The tadpole sum-integral in (22) can be separated into contributions from hard and soft regions, which we denote by the superscript (h) and (s) :

$$\oint_P \frac{1}{P^2 + m^2} = \oint_P^{(h)} \frac{1}{P^2 + m^2} + \oint_P^{(s)} \frac{1}{P^2 + m^2}. \quad (\text{A.1})$$

In the hard region, P^2 is of order T^2 , so we can expand in powers of m^2/P^2 :

$$\oint_P^{(h)} \frac{1}{P^2 + m^2} = \oint_P \left(\frac{1}{P^2} - \frac{m^2}{(P^2)^2} + \sum_{n=2}^{\infty} (-1)^n \frac{m^{2n}}{(P^2)^{n+1}} \right). \quad (\text{A.2})$$

The contribution from the soft region can be written

$$\oint_P^{(s)} \frac{1}{P^2 + m^2} = T \int_{\mathbf{p}} \frac{1}{p^2 + m^2}. \quad (\text{A.3})$$

One could accomplish the separation of the hard and soft momentum regions by introducing a separation scale Λ satisfying $m \ll \Lambda \ll 2\pi T$. The dependence on Λ would then cancel between (A.2) and (A.3). However, if we use dimensional regularization, we can get the

correct answer more easily by taking the limit $\Lambda \rightarrow 0$ in (A.2) and $\Lambda \rightarrow \infty$ in (A.3). The complete expansion of the sum-integral in powers of m/T is then just the sum of (A.2) and (A.3):

$$\oint_P \frac{1}{P^2 + m^2} = \oint_P \frac{1}{P^2} + TI_1 - m^2 \oint_P \frac{1}{(P^2)^2} + \sum_{n=2}^{\infty} (-1)^n m^{2n} \oint_P \frac{1}{(P^2)^{n+1}}, \quad (\text{A.4})$$

where I_1 is the momentum integral on the right side of (A.3).

The logarithmic sum-integral in (21) can also be separated into contributions from hard and soft regions, as in (A.1). In the hard contribution, we can expand in powers of m^2/P^2 :

$$\oint_P^{(h)} \log(P^2 + m^2) = \oint_P \left(\log P^2 + \frac{m^2}{P^2} - \frac{1}{2} \frac{m^4}{(P^2)^2} + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \frac{m^{2n}}{(P^2)^n} \right). \quad (\text{A.5})$$

The soft contribution is

$$\oint_P^{(s)} \log(P^2 + m^2) = T \int_{\mathbf{p}} \log(p^2 + m^2). \quad (\text{A.6})$$

The complete expansion of the sum-integral in powers of m/T is the sum of (A.5) and (A.6):

$$\begin{aligned} \oint_P \log(P^2 + m^2) &= \oint_P \log P^2 + m^2 \oint_P \frac{1}{P^2} + T(-I'_0) \\ &\quad - \frac{1}{2} m^4 \oint_P \frac{1}{(P^2)^2} + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} m^{2n} \oint_P \frac{1}{(P^2)^n}, \end{aligned} \quad (\text{A.7})$$

where $-I'_0$ is the momentum integral on the right side of (A.6).

b. Tadpole sum-integral

The simplest sum-integral with a momentum-dependent self-energy is the tadpole sum-integral, which appears in the expression (62) for the screening mass. The tadpole sum-integral can be separated into contributions from hard and soft regions, which we denote by a superscript (h) or (s) :

$$\oint_Q \frac{1}{Q^2 + \Pi(Q)} = \oint_Q^{(h)} \frac{1}{Q^2 + \Pi(Q)} + \oint_Q^{(s)} \frac{1}{Q^2 + \Pi(Q)}. \quad (\text{A.8})$$

In the hard region, Q^2 is of order T^2 while Π is of order m^2 , so we can expand in powers of $\Pi(Q^2)$. Inserting the expansion (65), we obtain

$$\begin{aligned} \rlap{-}\!\!\!\int_Q^{(h)} \frac{1}{Q^2 + \Pi(Q)} &= \rlap{-}\!\!\!\int_Q \left(\frac{1}{Q^2} - \frac{m^2}{(Q^2)^2} + \frac{m^4}{(Q^2)^3} \right) \\ &\quad - (g_0 \mu^{-\epsilon})^4 \rlap{-}\!\!\!\int_Q \frac{\Pi_{4,0}(Q) + \Pi_{4,1}(Q)}{(Q^2)^2} + \mathcal{O}(g^6). \end{aligned} \quad (\text{A.9})$$

The first term on the right side is proportional to $T^{2-2\epsilon}$. Taking m to be of order g , the others are suppressed by g^2 , g^4 , g^4 , and g^5 , respectively, in the limit $\epsilon \rightarrow 0$. The contribution from the soft region can be written

$$\rlap{-}\!\!\!\int_Q^{(s)} \frac{1}{Q^2 + \Pi(Q)} = T \int_{\mathbf{q}} \frac{1}{q^2 + m^2 + \sigma(q)}. \quad (\text{A.10})$$

In this region, q^2 is of order m^2 , which is the first term in the expansion (66) of $\Pi(0, p)$.

Expanding in powers of the higher terms, we have

$$\begin{aligned} \rlap{-}\!\!\!\int_Q^{(s)} \frac{1}{Q^2 + \Pi(Q)} &= T I_1 - (g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q) + \sigma_{4,0}(q)}{(q^2 + m^2)^2} \\ &\quad - (g_0 \mu^{-\epsilon})^8 T \int_{\mathbf{q}} \left(\frac{\sigma_{8,-4}(q)}{(q^2 + m^2)^2} - \frac{\sigma_{4,-2}^2(q)}{(q^2 + m^2)^3} \right) + \mathcal{O}(g^6). \end{aligned} \quad (\text{A.11})$$

The first term on the right side is proportional to $m^{1-2\epsilon} T$. In the limit $\epsilon \rightarrow 0$, it is of order g when m is of order gT . In this limit the other terms are of order g^3 , g^5 , g^5 , and g^5 , respectively. The error estimate $\mathcal{O}(g^6)$ on the sum-integrals (A.9) and (A.11) refers to their orders in g in the limit $\epsilon \rightarrow 0$.

The complete expression for the tadpole sum-integral is the sum of the hard and soft contributions in (A.9) and (A.11). The integrals and sum-integrals that do not involve the self energy functions $\Pi_{n,m}(Q)$ and $\sigma_{n,m}(q)$ are given in Appendices B and C.

c. Sunset sum-integral for hard P

The sunset sum-integral (60) depends on the external momentum P , and the expansion in powers of m/T is different for hard P and soft P . We first consider the case of hard P . There are 3 different sum-integration regions that must be considered. The momenta Q and R can both be hard, one can be hard and the other soft, or they both can be soft. We denote these regions by superscripts (hh) , (hs) , and (ss) , respectively.

In the (hh) region, we can expand in powers of the self-energy. Only the m^2 term in the self-energy (65) contributes up to order $g^4 T^2$:

$$\mathfrak{S}_{\text{sun}}^{(hh)}(P) = \not\!\!\!\int_{QR} \left(\frac{1}{Q^2 R^2 S^2} - \frac{3m^2}{(Q^2)^2 R^2 S^2} \right) + \mathcal{O}(g^4). \quad (\text{A.12})$$

We next consider the (hs) region for hard P . Taking into account that any of the three momenta Q , R , and $S = -(P + Q + R)$ can be soft, the sum-integral can be written

$$\mathfrak{S}_{\text{sun}}^{(hs)}(P) = 3T \int_{\mathbf{r}} \not\!\!\!\int_Q \frac{1}{Q^2 + \Pi(Q)} \frac{1}{r^2 + m^2 + \sigma(r)} \frac{1}{(P + Q)^2 + 2(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} + r^2 + \Pi(S)}, \quad (\text{A.13})$$

where $S = -(P + Q) - (0, \mathbf{r})$. The m/T expansion is obtained by expanding the sum-integrand around $1/[Q^2(r^2 + m^2)(P + Q)^2]$. The first few terms are

$$\begin{aligned} \mathfrak{S}_{\text{sun}}^{(hs)}(P) &= 3T I_1 \not\!\!\!\int_Q \frac{1}{Q^2(P + Q)^2} - 3T (I_0 + m^2 I_1) \not\!\!\!\int_Q \frac{1}{(Q^2)^2(P + Q)^2} \\ &\quad + \frac{12}{d} T (I_0 - m^2 I_1) \not\!\!\!\int_Q \frac{\mathbf{q}^2}{(Q^2)^3(P + Q)^2} \\ &\quad - 3(g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{r}} \frac{\sigma_{4,-2}(r)}{(r^2 + m^2)^2} \not\!\!\!\int_Q \frac{1}{Q^2(P + Q)^2} + \mathcal{O}(g^4). \end{aligned} \quad (\text{A.14})$$

With dimensional regularization, the integral I_0 vanishes. The factor of $1/d$ multiplying one of the sum-integrals comes from averaging over the angles of \mathbf{r} in d dimensions: $[(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r}]^2 \rightarrow (\mathbf{p} + \mathbf{q})^2 \mathbf{r}^2 / d$. The change of variables $Q \rightarrow -P - Q$ then puts the sum-integral over Q into the form shown.

Finally, we consider the (ss) region for hard P . Taking into account that the hard momentum can flow through any of the three propagators, the sum-integral can be written

$$\mathfrak{S}_{\text{sun}}^{(ss)}(P) = 3T^2 \int_{\mathbf{q}\mathbf{r}} \frac{1}{q^2 + m^2 + \sigma(q)} \frac{1}{r^2 + m^2 + \sigma(r)} \frac{1}{P^2 + 2\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) + (\mathbf{q} + \mathbf{r})^2 + \Pi(S)}, \quad (\text{A.15})$$

where $S = -P - (0, \mathbf{q} + \mathbf{r})$. The m/T expansion is obtained by expanding the sum-integral around $1/[(q^2 + m^2)(r^2 + m^2)P^2]$. The leading term is of order m^2 and all higher terms are of order g^4 or higher. Thus the (ss) term in the sum-integral can be written

$$\mathfrak{S}_{\text{sun}}^{(ss)}(P) = 3T^2 I_1^2 \frac{1}{P^2} + \mathcal{O}(g^4). \quad (\text{A.16})$$

The complete expression for the sunset sum-integral for hard momentum P , including all terms up to errors of order g^4 , is the sum of (A.12), (A.14), and (A.16).

d. Sunset sum-integral for soft P

We next carry out the m/T expansion for the sunset sum-integral with soft external momentum $P = (0, \mathbf{p})$, where \mathbf{p} is of order m . Again, there are three momentum regions to consider: (hh) , (hs) , and (ss) .

We first consider the (hh) region. The sum-integral can be written

$$\mathfrak{S}_{\text{sun}}^{(hh)}(0, \mathbf{p}) = \oint_{QR} \frac{1}{Q^2 + \Pi(Q)} \frac{1}{R^2 + \Pi(R)} \frac{1}{(Q+R)^2 + 2(\mathbf{q} + \mathbf{r}) \cdot \mathbf{p} + p^2 + \Pi(S)}, \quad (\text{A.17})$$

where $S = -(Q+R) - (0, \mathbf{p})$. The m/T expansion is obtained by expanding the sum-integrand around $1/[Q^2 R^2 (Q+R)^2]$. The first few terms are

$$\begin{aligned} \mathfrak{S}_{\text{sun}}^{(hh)}(0, \mathbf{p}) &= \oint_{QR} \frac{1}{Q^2 R^2 (Q+R)^2} - (p^2 + 3m^2) \oint_{QR} \frac{1}{(Q^2)^2 R^2 (Q+R)^2} \\ &\quad + \frac{4}{d} p^2 \oint_{QR} \frac{\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} + \mathcal{O}(g^4). \end{aligned} \quad (\text{A.18})$$

The first sum-integral on the right side vanishes.

We next consider the (hs) region. There must be one propagator with soft momentum. Taking into account that it can be any one of the three propagators, the sum-integral can be written

$$\mathfrak{S}_{\text{sun}}^{(hs)}(0, \mathbf{p}) = 3T \int_{\mathbf{r}} \oint_Q \frac{1}{Q^2 + \Pi(Q)} \frac{1}{r^2 + m^2 + \sigma(r)} \frac{1}{Q^2 + 2\mathbf{q} \cdot (\mathbf{p} + \mathbf{r}) + (\mathbf{p} + \mathbf{r})^2 + \Pi(S)}, \quad (\text{A.19})$$

where $S = -Q - (0, \mathbf{p} + \mathbf{r})$. The m/T expansion is obtained by expanding the sum-integrand around $1/[(Q^2)^2 (r^2 + m^2)]$. The first few terms are

$$\begin{aligned} \mathfrak{S}_{\text{sun}}^{(hs)}(0, \mathbf{p}) &= 3T I_1 \oint_Q \frac{1}{(Q^2)^2} - 3T [I_0 + (p^2 + m^2) I_1] \oint_Q \frac{1}{(Q^2)^3} \\ &\quad - 3(g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{r}} \frac{\sigma_{4,-2}(r)}{(r^2 + m^2)^2} \oint_Q \frac{1}{(Q^2)^2} \\ &\quad + \frac{12}{d} T [I_0 + (p^2 - m^2) I_1] \oint_Q \frac{\mathbf{q}^2}{(Q^2)^4} + \mathcal{O}(g^4). \end{aligned} \quad (\text{A.20})$$

The integrals I_0 vanish in dimensional regularization.

Finally, we consider the (ss) region. The sum-integral in this region can be written

$$\mathfrak{S}_{\text{sun}}^{(ss)} = T^2 \int_{\mathbf{qr}} \frac{1}{q^2 + m^2 + \sigma(q)} \frac{1}{r^2 + m^2 + \sigma(r)} \frac{1}{s^2 + m^2 + \sigma(s)}, \quad (\text{A.21})$$

where $\mathbf{s} = -(\mathbf{p} + \mathbf{q} + \mathbf{r})$. The m/T expansion is obtained by expanding in powers of $\sigma(q)$, $\sigma(r)$, and $\sigma(s)$. The first few terms in the expansion are

$$\mathfrak{S}_{\text{sun}}^{(ss)}(0, \mathbf{p}) = T^2 I_{\text{sun}}(p) - 3(g_0 \mu^{-\epsilon})^4 T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) + \mathcal{O}(g^4), \quad (\text{A.22})$$

where $I_{\text{sun}}(p)$ and $I_{\text{bub}}(p)$ are the sunset and bubble integrals defined in (C.10) and (C.9).

The complete expression for the sunset sum-integral at soft momentum $P = (0, \mathbf{p})$, including all terms up to errors of order g^4 , is the sum of (A.18), (A.20), and (A.22).

Evaluated at $p = im$, it reduces to

$$\begin{aligned} \mathfrak{S}_{\text{sun}}(0, \mathbf{p}) \Big|_{p=im} &= T^2 I_{\text{sun}}(im) + 3T I_1 \not\!\!\!\int_Q \frac{1}{(Q^2)^2} - \frac{24}{d} I_1 m^2 T \not\!\!\!\int_Q \frac{\mathbf{q}^2}{(Q^2)^4} \\ &\quad - 2m^2 \not\!\!\!\int_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} \\ &\quad - 3(g_0 \mu^{-\epsilon})^4 \left[T^2 \int_{\mathbf{q}} \frac{\sigma_{4,-2}(q)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) \Big|_{p=im} \right. \\ &\quad \left. + T \int_{\mathbf{r}} \frac{\sigma_{4,-2}(r)}{(r^2 + m^2)^2} \not\!\!\!\int_Q \frac{1}{(Q^2)^2} \right] + \mathcal{O}(g^4). \end{aligned} \quad (\text{A.23})$$

e. Sum-integrals in the thermodynamic potential

We now apply the m/T expansions to the sum-integrals that appear in the thermodynamic potential (57). The method should be clear from the example of the sunset diagram, so we only give the results for each momentum region.

For the logarithmic sum-integral in (57), the contributions from the hard and soft regions are

$$\not\!\!\!\int_P^{(h)} \log(P^2 + \Pi(P)) = \not\!\!\!\int_P \left[\log P^2 + \frac{m^2}{P^2} - \frac{m^4}{2(P^2)^2} + \frac{m^6}{3(P^2)^3} \right]$$

$$\begin{aligned}
& + (g_0 \mu^{-\epsilon})^4 \oint_P \left[\frac{\Pi_{4,0}(P) + \Pi_{4,1}(P) + \Pi_{4,2}(P) + \Pi_{4,3}(P)}{P^2} \right. \\
& \quad \left. - m^2 \frac{\Pi_{4,0}(P) + \Pi_{4,1}(P)}{(P^2)^2} \right] \\
& + (g_0 \mu^{-\epsilon})^8 \oint_P \frac{\Pi_{8,-2}(P) + \Pi_{8,-1}(P)}{P^2} + \mathcal{O}(g^8), \tag{A.24}
\end{aligned}$$

$$\begin{aligned}
\oint_P^{(s)} \log(P^2 + \Pi(P)) &= T(-I'_0) + (g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{p}} \frac{\sigma_{4,-2}(p) + \sigma_{4,0}(p)}{p^2 + m^2} \\
& + (g_0 \mu^{-\epsilon})^8 T \int_{\mathbf{p}} \left(\frac{\sigma_{8,-4}(p)}{p^2 + m^2} - \frac{\sigma_{4,-2}^2(p)}{2(p^2 + m^2)^2} \right) + \mathcal{O}(g^8). \tag{A.25}
\end{aligned}$$

For the other one-loop sum-integral in the thermodynamic potential, the hard and soft contributions are

$$\begin{aligned}
\oint_P^{(h)} \frac{\Pi(P)}{P^2 + \Pi(P)} &= \oint_P \left[\frac{m^2}{P^2} - \frac{m^4}{(P^2)^2} + \frac{m^6}{(P^2)^3} \right] \\
& + (g_0 \mu^{-\epsilon})^4 \oint_P \left[\frac{\Pi_{4,0}(P) + \Pi_{4,1}(P) + \Pi_{4,2}(P) + \Pi_{4,3}(P)}{P^2} \right. \\
& \quad \left. - 2m^2 \frac{\Pi_{4,0}(P) + \Pi_{4,1}(P)}{(P^2)^2} \right] \\
& + (g_0 \mu^{-\epsilon})^8 \oint_P \frac{\Pi_{8,-2}(P) + \Pi_{8,-1}(P)}{P^2} + \mathcal{O}(g^8), \tag{A.26}
\end{aligned}$$

$$\begin{aligned}
\oint_P^{(s)} \frac{\Pi(P)}{P^2 + \Pi(P)} &= m^2 T I_1 + (g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{p}} p^2 \frac{\sigma_{4,-2}(p) + \sigma_{4,0}(p)}{(p^2 + m^2)^2} \\
& + (g_0 \mu^{-\epsilon})^8 T \int_{\mathbf{p}} \left(p^2 \frac{\sigma_{8,-4}(p)}{(p^2 + m^2)^2} - p^2 \frac{\sigma_{4,-2}^2(p)}{(p^2 + m^2)^3} \right) + \mathcal{O}(g^8). \tag{A.27}
\end{aligned}$$

The two-loop sum-integral in (57) is the square of the tadpole sum-integral. It is multiplied by g_0^2 , so to get the free energy through order g^7 , we need the square of the tadpole integral to order g^5 . This is obtained by squaring the sum of (A.9) and (A.11), and keeping all terms to the desired order.

The basketball sum-integral in (57) involves a triple sum-integral, so there are 4 momentum regions: (sss) , (hss) , (hhs) , and (hhh) . The contribution from each of these regions is

$$\begin{aligned}
\mathfrak{S}_{\text{ball}}^{(hhh)} &= \oint_{PQR} \left(\frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} - \frac{4m^2}{(P^2)^2 Q^2 R^2 (P+Q+R)^2} \right) + \mathcal{O}(g^4), \tag{A.28} \\
\mathfrak{S}_{\text{ball}}^{(hhs)} &= 4T I_1 \oint_{QR} \frac{1}{Q^2 R^2 (Q+R)^2} - 4T (I_0 + 2m^2 I_1) \oint_{QR} \frac{1}{(Q^2)^2 R^2 (Q+R)^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{16}{d} T (I_0 - m^2 I_1) \not\int_{QR} \frac{\mathbf{q}^2}{(Q^2)^3 R^2 (Q+R)^2} \\
& - 4(g_0 \mu^{-\epsilon})^4 T \int_{\mathbf{p}} \frac{\sigma_{4,-2}(p)}{(p^2 + m^2)^2} \not\int_{QR} \frac{1}{Q^2 R^2 (Q+R)^2} + \mathcal{O}(g^4),
\end{aligned} \tag{A.29}$$

$$\mathfrak{S}_{\text{ball}}^{(hss)} = 6T^2 I_1^2 \not\int_Q \frac{1}{(Q^2)^2} + \mathcal{O}(g^4), \tag{A.30}$$

$$\mathfrak{S}_{\text{ball}}^{(sss)} = T^3 I_{\text{ball}} - 4(g_0 \mu^{-\epsilon})^4 T^3 \int_{\mathbf{p}} \frac{\sigma_{4,-2}(p) I_{\text{sun}}(p)}{(p^2 + m^2)^2} + \mathcal{O}(g^4), \tag{A.31}$$

where I_{ball} is the basketball integral given in (C.13). In (A.29), the first sum-integral and the integral I_0 vanish with dimensional regularization.

APPENDIX B: MASSLESS SUM-INTEGRALS

In this Appendix, we list the massless sum-integrals that appear in the calculation of the 3-loop Φ -derivable free energy through 7th order in g and m/T . Analytic expressions are given for all the sum-integrals with the exception of one 3-loop sum-integral, for which the constant term has not been evaluated.

In the imaginary-time formalism for thermal field theory, a boson has Euclidean 4-momentum $P = (p_0, \mathbf{p})$, with $P^2 = p_0^2 + \mathbf{p}^2$. The Euclidean energy p_0 has discrete values: $p_0 = 2\pi nT$, where n is an integer. Loop diagrams involve sums over p_0 and integrals over \mathbf{p} . With dimensional regularization, the integral is generalized to $d = 3 - 2\epsilon$ spacial dimensions. We define the dimensionally regularized sum-integral by

$$\not\int_P \equiv \left(\frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon T \sum_{p_0} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \tag{B.1}$$

where $d = 3 - 2\epsilon$ is the dimension of space and μ is an arbitrary momentum scale. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, μ coincides with the renormalization scale of the $\overline{\text{MS}}$ renormalization scheme.

The basic one-loop sum-integral is

$$\not\int_P \frac{1}{(P^2)^n} = \frac{\zeta(2n - 3 + 2\epsilon)}{8\pi^2} \frac{\Gamma(n - \frac{3}{2} + \epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} (e^\gamma \mu^2)^\epsilon (2\pi T)^{4-2n-2\epsilon}, \tag{B.2}$$

where γ is Euler's constant and $\zeta(z)$ is Riemann's zeta function. Most of the explicit one-loop sum-integrals required in our calculations can be derived from (B.2):

$$\oint_P \log P^2 = -\frac{\pi^2 T^4}{45} [1 + O(\epsilon)] , \quad (\text{B.3})$$

$$\oint_P \frac{1}{P^2} = \frac{T^2}{12} \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left[1 + \left(2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon + O(\epsilon^2) \right] , \quad (\text{B.4})$$

$$\oint_P \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2} \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left[\frac{1}{\epsilon} + 2\gamma + \left(\frac{\pi^2}{4} - 4\gamma_1 \right) \epsilon + O(\epsilon^2) \right] , \quad (\text{B.5})$$

$$\oint_P \frac{1}{(P^2)^3} = \frac{1}{(4\pi)^4 T^2} [2\zeta(3) + O(\epsilon)] , \quad (\text{B.6})$$

where $\hat{\mu} = \mu/(2\pi T)$. The number γ_1 is the first Stieltjes gamma constant defined by the equation

$$\zeta(1+z) = \frac{1}{z} + \gamma - \gamma_1 z + O(z^2) . \quad (\text{B.7})$$

We also need one other one-loop sum-integral:

$$\oint_P \frac{p_0^2}{(P^2)^4} = \frac{1}{(4\pi)^4 T^2} [\zeta(3) + O(\epsilon)] . \quad (\text{B.8})$$

The two-loop sum-integrals that are needed are

$$\oint_{PQ} \frac{1}{P^2 Q^2 (P+Q)^2} = 0 , \quad (\text{B.9})$$

$$\begin{aligned} \oint_{PQ} \frac{1}{(P^2)^2 Q^2 (P+Q)^2} &= \frac{1}{2(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left[\frac{1}{\epsilon^2} + (1+4\gamma) \frac{1}{\epsilon} \right. \\ &\quad \left. + \left(3 + \frac{\pi^2}{2} + 4\gamma + 4\gamma^2 - 8\gamma_1 \right) + O(\epsilon) \right] , \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} \oint_{PQ} \frac{p_0^2}{(P^2)^3 Q^2 (P+Q)^2} &= \frac{1}{8(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left[\frac{1}{\epsilon^2} + \left(\frac{9}{2} + 4\gamma \right) \frac{1}{\epsilon} \right. \\ &\quad \left. + \left(\frac{35}{4} + \frac{\pi^2}{2} + 18\gamma + 4\gamma^2 - 8\gamma_1 \right) + O(\epsilon) \right] . \end{aligned} \quad (\text{B.11})$$

The sum-integrals (B.10) and (B.11) can be evaluated using the methods of Ref. [6]. We will need two specific linear combinations of these sum-integrals:

$$\oint_{PQ} \frac{P^2 + (2/d)\mathbf{p}^2}{(P^2)^3 Q^2 (P+Q)^2} = \frac{3}{4(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left[\frac{1}{\epsilon^2} + \left(\frac{5}{6} + 4\gamma \right) \frac{1}{\epsilon} \right]$$

$$+ \left(\frac{89}{36} + \frac{\pi^2}{2} + \frac{10}{3}\gamma + 4\gamma^2 - 8\gamma_1 \right) + \mathcal{O}(\epsilon) \Big] , \quad (\text{B.12})$$

$$\oint_{PQ} \frac{P^2 - (4/d)\mathbf{p}^2}{(P^2)^3 Q^2 (P+Q)^2} = \frac{1}{4(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} \left[\frac{1}{\epsilon} + \left(\frac{19}{6} + 4\gamma \right) + \mathcal{O}(\epsilon) \right] . \quad (\text{B.13})$$

The three-loop sum-integrals that are required are

$$\oint_{PQR} \frac{1}{P^2 Q^2 R^2 S^2} = \frac{T^4}{24(4\pi)^2} \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left[\frac{1}{\epsilon} + \left(\frac{91}{15} + 8 \frac{\zeta'(-1)}{\zeta(-1)} - 2 \frac{\zeta'(-3)}{\zeta(-3)} \right) + \mathcal{O}(\epsilon) \right] , \quad (\text{B.14})$$

$$\oint_{PQR} \frac{1}{(P^2)^2 Q^2 R^2 S^2} = \frac{T^2}{8(4\pi)^4} \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} \left[\frac{1}{\epsilon^2} + \left(\frac{89}{6} + 4\gamma + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} \right] , \quad (\text{B.15})$$

where $S = P + Q + R$. The massless basketball sum-integral (B.14) was first evaluated analytically in Ref. [6]. The sum-integral (B.15) is more difficult. One way to evaluate it is to use the result for the massive basketball sum-integral in Ref. [20]. If that sum-integral is expanded in powers of m/T as in (D.11), the sum-integral (B.15) appears at order $(m/T)^2$. One can deduce the pole terms from the analytic expressions for the pole terms in the massive basketball sum-integral in Ref. [20]. We have not evaluated the constant term in this sum-integral.

The transcendental constants that appear in these sum-integrals are

$$\gamma = 0.57722 , \quad (\text{B.16})$$

$$\gamma_1 = -0.072816 , \quad (\text{B.17})$$

$$\zeta(3) = 1.20206 , \quad (\text{B.18})$$

$$\zeta'(-1)/\zeta(-1) = 1.98505 , \quad (\text{B.19})$$

$$\zeta'(-3)/\zeta(-3) = 0.64543 . \quad (\text{B.20})$$

APPENDIX C: INTEGRALS

In this Appendix, we list the integrals that appear in the calculation of the 3-loop Φ -derivable free energy through 7th order in g and m/T . The integrals that contribute through 5th order and those that contribute to the pole terms at 6th and 7th order are evaluated

analytically. Some of the integrals that contribute to finite terms at 6th and 7th order are evaluated numerically.

In a Euclidean field theory in 3 space dimensions, loop diagrams involve integrals over 3-momenta. With dimensional regularization, the integral is generalized to $d = 3 - 2\epsilon$ spacial dimensions. We define the measure for the dimensionally regularized integral by

$$\int_{\mathbf{p}} \equiv \left(\frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}. \quad (\text{C.1})$$

If renormalization is accomplished by the minimal subtraction of poles in ϵ , then μ is the renormalization scale in the $\overline{\text{MS}}$ scheme.

The basic one-loop integral for a boson with mass m is

$$I_n = \int_{\mathbf{p}} \frac{1}{(p^2 + m^2)^n}. \quad (\text{C.2})$$

Differentiating with respect to n and then setting $n = 0$, we obtain the integral

$$-I'_0 = \int_{\mathbf{p}} \log(p^2 + m^2). \quad (\text{C.3})$$

The analytic expression for the integral (C.2) is

$$I_n = \frac{1}{8\pi} \frac{\Gamma(n - \frac{3}{2} + \epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} \left(e^\gamma \mu^2 \right)^\epsilon m^{3-2n-2\epsilon}. \quad (\text{C.4})$$

The explicit one-loop integrals required in our calculations can be derived from (C.4):

$$I_0 = 0, \quad (\text{C.5})$$

$$-I'_0 = -\frac{m^3}{6\pi} \left(\frac{\mu}{2m} \right)^{2\epsilon} \left[1 + \frac{8}{3}\epsilon + O(\epsilon^2) \right], \quad (\text{C.6})$$

$$I_1 = -\frac{m}{4\pi} \left(\frac{\mu}{2m} \right)^{2\epsilon} \left[1 + 2\epsilon + \left(4 + \frac{\pi^2}{4} \right) \epsilon^2 + O(\epsilon^3) \right], \quad (\text{C.7})$$

$$I_2 = \frac{1}{8\pi m} \left(\frac{\mu}{2m} \right)^{2\epsilon} \left[1 + O(\epsilon^2) \right]. \quad (\text{C.8})$$

The multi-loop integrals that are needed can be conveniently expressed in terms of the bubble and sunset integrals defined by

$$I_{\text{bub}}(p) = \int_{\mathbf{q}} \frac{1}{q^2 + m^2} \frac{1}{(\mathbf{p} + \mathbf{q})^2 + m^2}, \quad (\text{C.9})$$

$$I_{\text{sun}}(p) = \int_{\mathbf{q}, \mathbf{r}} \frac{1}{q^2 + m^2} \frac{1}{r^2 + m^2} \frac{1}{(\mathbf{p} + \mathbf{q} + \mathbf{r})^2 + m^2}. \quad (\text{C.10})$$

We need the analytic continuation of the sunset integral to $p = im$. It is evaluated in Ref. [2] through order ϵ^0 , but we also need the order ϵ term:

$$I_{\text{sun}}(im) = \frac{1}{4(4\pi)^2} \left(\frac{\mu}{2m} \right)^{4\epsilon} \left[\frac{1}{\epsilon} + (6 - 8 \log 2) + \left(24 - \frac{7\pi^2}{6} - 2\gamma + 8\gamma^2 - 48 \log 2 + 8 \log^2 2 \right) \epsilon + O(\epsilon^2) \right]. \quad (\text{C.11})$$

The three-loop basketball integral can be expressed as

$$I_{\text{ball}} = \int_{\mathbf{p}} \frac{I_{\text{sun}}(p)}{p^2 + m^2}. \quad (\text{C.12})$$

This integral is also evaluated in Ref. [2]:

$$I_{\text{ball}} = -\frac{m}{(4\pi)^3} \left(\frac{\mu}{2m} \right)^{6\epsilon} \left[\frac{1}{\epsilon} + (8 - 4 \log 2) + O(\epsilon) \right]. \quad (\text{C.13})$$

The other three-loop integral is

$$\int_{\mathbf{p}} \frac{I_{\text{sun}}(p) - I_{\text{sun}}(im)}{(p^2 + m^2)^2} = -\frac{1}{2(4\pi)^3 m} [(1 - \log 2) + O(\epsilon)]. \quad (\text{C.14})$$

This can be obtained by differentiating (C.13) with respect to m^2 and also using (C.8) and (C.11). The integral (C.14) is convergent for $d = 3$, so it can also be calculated directly without using dimensional regularization. In the limit $\epsilon \rightarrow 0$, the function in the numerator is

$$I_{\text{sun}}(p) - I_{\text{sun}}(im) \longrightarrow -\frac{1}{(4\pi)^2} \left[\frac{3m}{p} \text{atan} \frac{p}{3m} + \frac{1}{2} \log \frac{p^2 + 9m^2}{64m^2} \right]. \quad (\text{C.15})$$

The 4-loop integral that is needed is

$$\int_{\mathbf{q}} \frac{I_{\text{sun}}(q) - I_{\text{sun}}(im)}{(q^2 + m^2)^2} I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) \Big|_{p=im} = \frac{1}{(4\pi)^4 m^2} [-0.022048 + O(\epsilon)]. \quad (\text{C.16})$$

This integral can be evaluated directly in $d = 3$ dimensions. The bubble integral is

$$I_{\text{bub}}(|\mathbf{p} + \mathbf{q}|) \longrightarrow \frac{1}{4\pi|\mathbf{p} + \mathbf{q}|} \text{atan} \frac{|\mathbf{p} + \mathbf{q}|}{2m}. \quad (\text{C.17})$$

After averaging over the angles of \mathbf{q} , we can set $p = im$. Inserting the resulting function into (C.16) along with (C.15), we obtain a one-dimensional integral that can be evaluated numerically.

Finally there are several 5-loop integrals that are needed. Two of them are very simple:

$$\int_{\mathbf{p}} \frac{[I_{\text{sun}}(p) - I_{\text{sun}}(im)]^2}{(p^2 + m^2)^2} = \frac{1}{(4\pi)^6 m} [2.52519 + O(\epsilon)], \quad (\text{C.18})$$

$$\int_{\mathbf{p}} \frac{[I_{\text{sun}}(p) - I_{\text{sun}}(im)]^2}{(p^2 + m^2)^3} = \frac{1}{(4\pi)^6 m^3} [0.012004 + O(\epsilon)]. \quad (\text{C.19})$$

They can be evaluated directly in $d = 3$ dimensions by inserting the expression (C.15) for the function in the numerator. The other 5-loop integral is

$$\int_{\mathbf{p}} \frac{I_{\text{sun}}(p) - I_{\text{sun}}(im)}{(p^2 + m^2)^2} \frac{d}{dm^2} I_{\text{sun}}(p) = \frac{1}{(4\pi)^6 m^3} [0.344713 + O(\epsilon)]. \quad (\text{C.20})$$

In the limit $\epsilon \rightarrow 0$, the last factor in the integrand reduces to

$$\frac{d}{dm^2} I_{\text{sun}}(p) \longrightarrow -\frac{1}{(4\pi)^2} \left[\frac{3}{2mp} \text{atan} \frac{p}{3m} \right]. \quad (\text{C.21})$$

The integral in (C.20) can be evaluated directly in $d = 3$ dimensions by inserting the expressions (C.15) and (C.21) for the functions in the numerator.

APPENDIX D: MASSIVE SUM-INTEGRALS

To calculate the m/T expansion of the thermodynamic potential, we need to expand several massive sum-integrals in powers of m/T . The m/T expansions can be obtained from those in Appendix A simply by omitting terms containing the self-energy functions $\Pi_{n,k}(P)$ and $\sigma_{n,k}(p)$. The remaining terms in the m/T expansion involve the sum-integrals in Appendix B and the integrals in Appendix C.

The two massive one-loop sum-integrals that we need are

$$-\mathcal{I}'_0 = \oint_P \log(P^2 + m^2), \quad (\text{D.1})$$

$$\mathcal{I}_1 = \oint_P \frac{1}{P^2 + m^2}. \quad (\text{D.2})$$

Their m/T expansions are given in (A.4) and (A.7):

$$-\mathcal{I}'_0 = T(-I'_0) + \oint_P \left[\log P^2 + \frac{m^2}{P^2} - \frac{m^4}{2(P^2)^2} + \frac{m^6}{3(P^2)^3} \right] + \mathcal{O}(m^8), \quad (\text{D.3})$$

$$\mathcal{I}_1 = TI_1 + \oint_P \left[\frac{1}{P^2} - \frac{m^2}{(P^2)^2} + \frac{m^4}{(P^2)^3} \right] + \mathcal{O}(m^6). \quad (\text{D.4})$$

Inserting the sum-integrals from Appendix B and the integrals from Appendix C, we get

$$-\mathcal{I}'_0 = -\frac{\pi^2}{45} T^4 \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left\{ 1 - 15\hat{m}^2 + 60\hat{m}^3 + \frac{45}{2} \left(\frac{1}{\epsilon} + 2\gamma \right) \hat{m}^4 - \frac{15}{2} \zeta(3) \hat{m}^6 \right\} + \mathcal{O}(m^8), \quad (\text{D.5})$$

$$\mathcal{I}_1 = \frac{1}{12} T^2 \left(\frac{\hat{\mu}}{2} \right)^{2\epsilon} \left\{ \left[1 + \left(2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon \right] - 6 [1 + (-2 \log \hat{m} + 2) \epsilon] \hat{m} - 3 \left(\frac{1}{\epsilon} + 2\gamma \right) \hat{m}^2 + \frac{3}{2} \zeta(3) \hat{m}^4 \right\} + \mathcal{O}(m^6), \quad (\text{D.6})$$

where $\hat{m} = m/(2\pi T)$ and $\hat{\mu} = \mu/(2\pi T)$. The errors in the coefficients of \hat{m}^n are one order higher in ϵ than the last term shown explicitly.

The screening sum-integral for a particle of mass m is

$$\mathcal{I}_{\text{screen}} = \oint_{QR} \frac{1}{(Q^2 + m^2)(R^2 + m^2)((P + Q + R)^2 + m^2)} \Big|_{P=(0, \mathbf{p}), p=im}. \quad (\text{D.7})$$

The m/T expansion is obtained by omitting the terms in (A.23) involving $\sigma_{4,-2}(p)$:

$$\begin{aligned} \mathcal{I}_{\text{screen}} &= T^2 I_{\text{sun}}(im) + 3TI_1 \oint_Q \frac{1}{(Q^2)^2} \\ &\quad - 2m^2 \oint_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{(Q^2)^3 R^2 (Q + R)^2} \\ &\quad - \frac{24}{d} I_1 m^2 T \oint_Q \frac{\mathbf{q}^2}{(Q^2)^4} + \mathcal{O}(m^4). \end{aligned} \quad (\text{D.8})$$

The sum-integrals and integrals are given in Appendices B and C. Keeping finite terms through order m and divergent terms through order m^3 , we get

$$\begin{aligned}
\mathcal{I}_{\text{screen}} = \frac{1}{4(4\pi)^2} T^2 \left(\frac{\hat{\mu}}{2} \right)^{4\epsilon} & \left\{ \left[\frac{1}{\epsilon} + (-4 \log \hat{m} + 6 - 8 \log 2) \right] \right. \\
& - 6 \left[\frac{1}{\epsilon} + (-2 \log \hat{m} + 2 + 2\gamma) \right] \hat{m} \\
& \left. - \frac{3}{2} \left[\frac{1}{\epsilon^2} + \left(\frac{5}{6} + 4\gamma \right) \frac{1}{\epsilon} \right] \hat{m}^2 \right\}. \tag{D.9}
\end{aligned}$$

The massive basketball sum-integral is

$$\mathcal{I}_{\text{ball}} = \not\!\!\!\int_{PQR} \frac{1}{(P^2 + m^2)(Q^2 + m^2)(R^2 + m^2)(S^2 + m^2)}, \tag{D.10}$$

where $S = P + Q + R$. The m/T expansion of this sum-integral is obtained by dropping the terms involving $\sigma_{4,-2}(p)$ in the sum of (A.28)-(A.31):

$$\begin{aligned}
\mathcal{I}_{\text{ball}} = & \not\!\!\!\int_{PQR} \left(\frac{1}{P^2 Q^2 R^2 S^2} - \frac{4m^2}{(P^2)^2 Q^2 R^2 S^2} \right) \\
& + T^3 I_{\text{ball}} + 6T^2 I_1^2 \not\!\!\!\int_Q \frac{1}{(Q^2)^2} \\
& - 8Tm^2 I_1 \not\!\!\!\int_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{(Q^2)^3 R^2 (Q + R)^2} + \mathcal{O}(m^4). \tag{D.11}
\end{aligned}$$

The sum-integrals and integrals are given in Appendices B and C. Keeping finite terms through order m and divergent terms through order m^3 , we get

$$\begin{aligned}
\mathcal{I}_{\text{ball}} = \frac{T^4}{24(4\pi)^2} \left(\frac{\hat{\mu}}{2} \right)^{6\epsilon} & \left\{ \left[\frac{1}{\epsilon} + \frac{91}{15} + 8 \frac{\zeta'(-1)}{\zeta(-1)} - 2 \frac{\zeta'(-3)}{\zeta(-3)} \right] \right. \\
& - 12 \left[\frac{1}{\epsilon} - 6 \log \hat{m} + 8 - 4 \log 2 \right] \hat{m} \\
& - 3 \left[\frac{1}{\epsilon^2} + \left(\frac{17}{6} + 4\gamma + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} \right] \hat{m}^2 \\
& \left. + 18 \left[\frac{1}{\epsilon^2} + \left(-2 \log \hat{m} + \frac{17}{6} + 4\gamma \right) \frac{1}{\epsilon} \right] \hat{m}^3 \right\}. \tag{D.12}
\end{aligned}$$

REFERENCES

- [1] R.R. Parwani and H. Singh, Phys. Rev. **D51**, 4518 (1995).
- [2] E. Braaten and A. Nieto, Phys. Rev. **D51**, 6990 (1995).
- [3] C. Coriano and R.R. Parwani, Phys. Rev. Lett. **73**, 2398 (1994).
- [4] R.R. Parwani, Phys. Lett. **B334**, 420 (1994); [Erratum-ibid. B **342**, 454 (1994)]; R.R. Parwani and C. Coriano, Nucl. Phys. **B434**, 56 (1995).
- [5] J.O. Andersen, Phys. Rev. **D53**, 7286 (1996).
- [6] P. Arnold and C. Zhai, Phys. Rev. **D50**, 7603 (1994); Phys. Rev. **D51**, 1906 (1995).
- [7] B. Kastening and C. Zhai, Phys. Rev. **D52**, 7232 (1995).
- [8] E. Braaten and A. Nieto, Phys. Rev. Lett. **76**, 1417 (1996); Phys. Rev. **D53**, 3421 (1996).
- [9] F. Karsch, A. Patkós, and P. Petreczky, Phys. Lett. **B401**, 69 (1997).
- [10] J.M. Luttinger and J.C. Ward, Phys. Rev. **118**, 1417 (1960).
- [11] G. Baym, Phys. Rev. **127**, 1391 (1962).
- [12] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).
- [13] B. Vanderheyden and G. Baym, J. Stat. Phys. **93**, 843 (1998).
- [14] J. P. Blaizot, E. Iancu and A. Rebhan, Phys. Rev. Lett. **83**, 2906 (1999); Phys. Lett. B **470**, 181 (1999); Phys. Rev. D **63**, 065003 (2001).
- [15] A. Peshier, Phys. Rev. D **63**, 105004 (2001).
- [16] Y. B. Ivanov, J. Knoll and D. N. Voskresensky, Nucl. Phys. A **657**, 413 (1999).
- [17] J. Berges, hep-ph/0105311.
- [18] A. Peshier, B. Kämpfer, O. P. Pavlenko, and G. Soff, Phys. Rev. **D54**, 2399 (1996).

- [19] H. Kleinert et al., Phys. Lett. **B272**, 39 (1990); **319**, 545(E) (1993).
- [20] J.O. Andersen, E. Braaten, and M. Strickland, Phys. Rev. **D62**, 045004 (2000).
- [21] J. O. Andersen, E. Braaten and M. Strickland, Phys. Rev. D **63**, 105008 (2001);
J. O. Andersen and M. Strickland, hep-ph/0105214.